

# MODULES FOR YOKONUMA-TYPE HECKE ALGEBRAS

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**ABSTRACT.** This paper describes the module categories for a family of generic Hecke algebras, called Yokonuma-type Hecke algebras. Yokonuma-type Hecke algebras specialize both to the group algebras of the complex reflection groups  $G(r, 1, n)$  and to the convolution algebras of  $(B', B')$ -double cosets in the group algebras of finite general linear groups, for certain subgroups  $B'$  consisting of upper triangular matrices. In particular, complete sets of inequivalent, irreducible modules for semisimple specializations of Yokonuma-type Hecke algebras are constructed.

## 1. INTRODUCTION

1.1. Suppose  $n$  is a positive integer and  $q$  is a prime power. Let  $G = GL_n(\mathbb{F}_q)$  be the group of all invertible  $n \times n$  matrices with entries in the finite field  $\mathbb{F}_q$  and let  $U$  denote the subgroup of  $G$  consisting of all unipotent upper triangular matrices. The natural action of  $G$  on the set of cosets  $G/U$  determines the permutation representation  $\text{Ind}_U^G(1_U)$  of  $G$  (over  $\mathbb{C}$ ). The endomorphism algebra of this representation is anti-isomorphic to the subalgebra  $\mathcal{H}_{\mathbb{C}}(G, U)$  of the group algebra  $\mathbb{C}[G]$  consisting of functions that are constant on  $(U, U)$ -double cosets. For a finite Chevalley group  $G'$  with maximal unipotent group  $U'$ , Yokonuma [13] gave a presentation of  $\mathcal{H}_{\mathbb{C}}(G', U')$

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and described some of its structure. When the prime power  $q$  is replaced by an indeterminate  $\mathbf{q}$ , Yokonuma's presentation (extended to  $G$ , see [9]) defines an algebra over the polynomial ring  $\mathbb{Z}[\mathbf{q}]$ . We call these algebras one-parameter Yokonuma-Hecke algebras. The irreducible  $\mathcal{H}_{\mathbb{C}}(G, U)$ -modules have been constructed and classified by Thiem [12] using a weight space-type decomposition. Using combinatorial methods, Chlouveraki and Poulain d'Andecy [3] have constructed and classified the irreducible modules for one-parameter Yokonuma-Hecke algebras.

Now suppose that  $a$  and  $b$  are relatively prime positive integers such that  $q-1 = ab$ . Then the multiplicative group of  $\mathbb{F}_q$  (a finite cyclic group of order  $q-1$ ) has a unique factorization as  $F_a F_b$ , where  $|F_a| = a$  and  $|F_b| = b$ . Let  $H_a$  (resp.  $H_b$ ) denote the subgroup of  $G$  consisting of diagonal matrices with entries in  $F_a$  (resp.  $F_b$ ). For example, if  $l$  is a prime that does not divide either  $q$  or  $n!$  and  $a$  is the  $l$ -part of  $q-1$ , then  $H_a$  is a Sylow  $l$ -subgroup of  $G$ . Set  $B_a = H_a U$ . Notice that if  $a = 1$ , then  $B_a = U$  and if  $a = q-1$ , then  $B_a = B$  is the Borel subgroup of upper triangular matrices in  $G$ . Yokonuma's results in [13] can be used to describe the algebra  $\mathcal{H}_{\mathbb{C}}(G, B_a)$  as a subalgebra of  $\mathcal{H}_{\mathbb{C}}(G, U)$ . A presentation of the algebra  $\mathcal{H}_{\mathbb{C}}(G, B_a)$  was given in [1, §2]. When the prime power  $q$  and the divisor  $a$  are replaced by indeterminates  $\mathbf{q}$  and  $\mathbf{a}$ , the presentation in [1, §3] defines an algebra over the polynomial ring  $\mathbb{Z}[\mathbf{q}, \mathbf{a}]$ . We call these algebras Yokonuma-type Hecke algebras or two-variable Yokonuma-Hecke algebras.

The Iwahori-Hecke algebra of  $G$  is a  $\mathbb{Z}[\mathbf{q}]$ -algebra that specializes both to the centralizer algebra  $\mathcal{H}_{\mathbb{C}}(G, B)$  and to the group algebra  $\mathbb{C}[W]$ , where  $W$  is the group of permutation matrices in  $G$ . Similarly, Yokonuma-type Hecke algebras specialize both to the centralizer algebra  $\mathcal{H}_{\mathbb{C}}(G, B_a)$  and to the group algebra  $\mathbb{C}[WH_b]$ . The group  $WH_b$  is abstractly isomorphic to the complex reflection group denoted by  $G(b, 1, n)$ .

The main results in this paper are a “block” decomposition of the module category of a Yokonuma-type Hecke algebra in §2, and an explicit construction and classification of the simple modules for semisimple specializations of these algebras in §3. Our approach is based on Thiem's, but even for the algebras considered in [12] we give a more detailed description of the structure of these module categories and finer information about the structure of the simple modules.

Yokonuma-type Hecke algebras have been studied in several papers using different presentations. To help the reader, we explain in §4 the precise relationships between the presentation used by Chlouveraki and Poulain d'Andecy [3, §2.1], the presentation used by Jacon and Poulain d'Andecy [8, §2.3], and the presentation in this paper. The relationship between Yokonuma's original presentation of  $\mathcal{H}_{\mathbb{C}}(G, U)$  and the presentation found by Juyumaya [10, §2] is also sketched.

The results in this paper both supplement and complement results about the structure of  $\mathcal{H}$  and  $\mathcal{H}$ -modules given in [3] and [8]. In particular, the constructions in §2 and §3 lead to a uniform approach to the different constructions of simple modules given in these two papers.

Let  $\mathcal{H}$  be the Yokonuma-type Hecke algebra defined in 2.1 below, with scalars extended to  $\mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}, \mathbf{a}]$ . As described in §4,  $\mathcal{H}$  is the algebra  $Y_{d,n}$  defined in [8,

§2.3] and the  $\mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$ -algebra  $Y_{d,n}(\mathbf{q})$  in [3, §2.1] is a specialization of  $\mathcal{H}$ . With the notation introduced in the next section, there is a direct sum decomposition

$$(a) \quad \mathcal{H} \cong \bigoplus_{\alpha \in \mathcal{C}} \mathcal{H}e_{\alpha},$$

where  $\mathcal{H}e_{\alpha}$  is a two-sided ideal in  $\mathcal{H}$  and the index  $\alpha$  determines a Young subgroup  $\Sigma_{\alpha}$  of the symmetric group  $\Sigma_n$ . Lusztig [11, §34] and Jacon and Poulain d'Andecy [8, §3] construct explicit isomorphisms between  $\mathcal{H}e_{\alpha}$  and the matrix ring  $M_{r_{\alpha}}(\mathcal{I}^{\alpha})$ , where  $\mathcal{I}^{\alpha}$  is the Iwahori-Hecke algebra of  $\Sigma_{\alpha}$ . The block decomposition given in §2 may be viewed as a categorical framework underlying these isomorphisms.

The simple  $\mathbb{C}(\mathbf{q})Y_{d,n}(\mathbf{q})$ -modules have been constructed by Chlouveraki and Poulain d'Andecy [3, §5] by defining an action of the generators of  $\mathbb{C}(\mathbf{q})Y_{d,n}(\mathbf{q})$  on a basis and checking directly that the defining relations of  $\mathbb{C}(\mathbf{q})Y_{d,n}(\mathbf{q})$  hold. Somewhat more directly, as observed in [8, §4.1], it follows from the decomposition (a) and the isomorphisms  $\mathcal{H}e_{\alpha} \cong M_{r_{\alpha}}(\mathcal{I}^{\alpha})$  that every simple  $\mathcal{H}$ -module is the space of column vectors of size  $r_{\alpha}$  with entries in  $M$ , where  $M$  is a simple  $\mathcal{I}^{\alpha}$ -module.

In analogy with the so-called method of little groups [4, 11.8] (which applies to the group  $G(b, 1, n)$ ) it follows from block decomposition in §2 that every simple  $\mathcal{H}$ -module is of the form  $\mathcal{H} \otimes_{\mathcal{H}_{\alpha}} M$ , where  $\mathcal{H}_{\alpha}$  is a subalgebra of  $\mathcal{H}$  that is isomorphic to  $\mathcal{I}^{\alpha}$  and  $M$  is a simple  $\mathcal{H}_{\alpha}$ -module. As an  $\mathcal{H}_{\alpha}$ -module,  $\mathcal{H}$  is free with rank  $r_{\alpha}$ . In this way we obtain a coordinate-free construction of the simple modules in [8]. There are several constructions of  $\mathcal{I}^{\alpha}$ -modules that yield a complete set of simple modules for any semisimple specialization of  $\mathcal{I}^{\alpha}$ . Using Hoefsmit's construction [7] and the induction result from §2, in §3 a family of  $\mathcal{H}$ -modules is defined that specializes to the modules for one-parameter Yokonuma-Hecke algebras constructed in [3, §5]. The construction in §3 is valid more generally for Yokonuma-type Hecke algebras and gives more information about these modules than is immediately evident from their description in [3, §5]. A new feature of the construction in this paper is that it highlights how the natural action of  $W$  on the set of  $b$ -tableau is reflected in the module structure of these  $\mathcal{H}$ -modules, see Proposition 3.9 and Theorem 3.17.

**1.2. Notation.** For a positive integer  $l$ ,  $[l]$  denotes the set  $\{1, \dots, l\}$  and  $\Sigma_l$  denotes the group of permutations of  $[l]$ .

## 2. A BLOCK DECOMPOSITION OF $\mathcal{H}$

In this section we begin with the definition of the Yokonuma-type Hecke algebra  $\mathcal{H}$  from [1], then we describe the algebra structure of  $\mathcal{H}$  and give some specific specializations. The main result is a block decomposition of the category of  $\mathcal{H}$ -modules.

**2.1.** Suppose from now on that  $n$  and  $b$  are positive integers and that  $\mathbf{q}$  and  $\mathbf{a}$  are indeterminates. Define  $\mathcal{H}_{b,n}$  to be the unital  $\mathbb{Z}[\mathbf{q}, \mathbf{a}]$ -algebra with generators

$$T_1, \dots, T_n, R_1, \dots, R_{n-1}$$

and relations

$$\begin{aligned}
(\mathbf{r}_1) \quad & T_j^b = 1 && \text{for } j \in [n], \\
(\mathbf{r}_2) \quad & T_j T_{j'} = T_{j'} T_j && \text{for } j, j' \in [n], \\
(\mathbf{r}_3) \quad & T_j R_i = R_i T_{s_i(j)} && \text{for } j \in [n] \text{ and } i \in [n-1], \\
(\mathbf{r}_4) \quad & R_i R_{i'} = R_{i'} R_i && \text{for } i, i' \in [n-1] \text{ with } |i - i'| > 1, \\
(\mathbf{r}_5) \quad & R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1} && \text{for } i \in [n-2], \text{ and} \\
(\mathbf{r}_6) \quad & R_i^2 = \mathbf{q} + \mathbf{a} T_i^{(b^2-b)/2} E_i R_i && \text{for } i \in [n-1],
\end{aligned}$$

where  $s_i = (i \ i+1)$  is the transposition in the symmetric group  $\Sigma_n$  that switches  $i$  and  $i+1$  and

$$E_i = \sum_{k=0}^{b-1} T_i^k T_{i+1}^{-k}.$$

If  $b$  is odd, then  $T_i^{(b^2-b)/2} = 1$ , and if  $b$  is even, then  $T_i^{(b^2-b)/2} = T_i^{b/2}$  is a square root of 1. Notice that  $\mathcal{H}_{1,n}$  is the Iwahori-Hecke algebra of type  $A_{n-1}$  with parameters  $\mathbf{q}$  and  $\mathbf{a}$ .

2.2. Let  $\mu_b = \langle \zeta \rangle$  be the group of  $b^{\text{th}}$  roots of unity in  $\mathbb{C}$  and let  $D$  denote the group of diagonal matrices in  $\text{GL}_n(\mathbb{C})$  with diagonal entries in  $\mu_b$ . Suppose  $d \in D$  and that the  $(j, j)$ -entry of  $d$  is  $\zeta^{p_j}$  for  $j \in [n]$ . Define

$$t_d = T_1^{p_1} \cdots T_n^{p_n} \in \mathcal{H}_{b,n}.$$

Then if  $\mathcal{D}$  denotes the subalgebra of  $\mathcal{H}_{b,n}$  generated by  $\{T_1, \dots, T_n\}$ , it follows from relations  $(\mathbf{r}_1)$  and  $(\mathbf{r}_2)$  that the rule  $d \mapsto t_d$  defines a  $\mathbb{Z}[\mathbf{q}, \mathbf{a}]$ -algebra isomorphism between the group algebra  $\mathbb{Z}[\mathbf{q}, \mathbf{a}][D]$  and  $\mathcal{D}$ .

2.3. Let  $W$  denote the group of permutation matrices in  $\text{GL}_n(\mathbb{C})$ . The matrices in  $W$  permute the standard basis vectors  $v_1, \dots, v_n$  of  $\mathbb{C}^n$ , where  $v_i$  is the column vector in  $\mathbb{C}^n$  whose only non-zero entry is a 1 in the  $i^{\text{th}}$  coordinate. We use the bijection  $v_i \leftrightarrow i$  between  $\{v_1, \dots, v_n\}$  and  $\{1, \dots, n\}$  to identify permutation matrices with permutations. Precisely, for  $w \in W$ ,  $w$  also denotes the permutation in the symmetric group  $\Sigma_n$  defined by

$$wv_i = v_{w(i)}.$$

Whether  $w \in W$  denotes a matrix or a permutation will always be clear from context.

For  $i \in [n-1]$  let  $s_i$  denote the permutation matrix in  $W$  obtained from the identity matrix by interchanging rows  $i$  and  $i+1$  and set  $\mathcal{S} = \{s_1, \dots, s_{n-1}\}$ . (Note that we considered  $s_i$  as a permutation in relation  $(\mathbf{r}_3)$ .) Then  $\mathcal{S}$  is a set of Coxeter generators for  $W$  and so determines a length function on  $W$  and the notion of a reduced expression of an element of  $W$ . Suppose  $w \in W$  and  $s_{i_1} \cdots s_{i_k}$  is a reduced expression for  $w$ . Define

$$t_w = R_{i_1} \cdots R_{i_k} \in \mathcal{H}_{b,n}.$$

It follows from relations  $(\mathbf{r}_4)$ ,  $(\mathbf{r}_5)$ , and Matsumoto's Theorem that  $t_w$  is well-defined.

2.4. The group  $W$  normalizes  $D$  and the semidirect product  $WD$  is the complex reflection group denoted by  $G(b, 1, n)$ . For  $x = wd \in WD$ , define

$$t_x = t_w t_d \in \mathcal{H}_{b,n}.$$

The element  $t_x$  is well-defined because  $W \cap D = 1$ . Extend the length function  $\ell$  on  $W$  to all of  $WD$  by defining  $\ell(x) = \ell(w)$  when  $x = wd$ . The next theorem is proved in [1, §3].

**Theorem 2.5.** *The set  $\{t_x \mid x \in WD\}$  is an  $\mathbb{Z}[\mathbf{q}, \mathbf{a}]$ -basis of  $\mathcal{H}_{b,n}$ . For  $x$  in  $WD$ ,  $d$  in  $D$ , and  $s = s_i$  in  $\mathcal{S}$  the following relations hold:*

$$\begin{aligned} t_x t_d &= t_{xd} \\ t_d t_x &= t_{dx} \\ t_s t_x &= \begin{cases} t_{sx} & \text{if } \ell(sx) > \ell(x) \\ \mathbf{q} t_{sx} + \mathbf{a} T_i^{(b^2-b)/2} E_i t_x & \text{if } \ell(sx) < \ell(x) \end{cases} \\ t_x t_s &= \begin{cases} t_{xs} & \text{if } \ell(xs) > \ell(x) \\ \mathbf{q} t_{xs} + \mathbf{a} t_x T_i^{(b^2-b)/2} E_i & \text{if } \ell(xs) < \ell(x). \end{cases} \end{aligned}$$

2.6. **Specialization.** A specialization of a commutative ring with identity  $R$  is a ring homomorphism  $f: R \rightarrow k$ , where  $k$  is a commutative ring with identity and  $f(1) = 1$ .

Suppose  $H$  is an  $R$ -algebra and  $f: R \rightarrow k$  is a specialization. Then  $f$  determines an  $R$ -module structure on  $k$  and we may form the specialized algebra

$${}_f H = k \otimes_R H.$$

If  $V$  is an  $H$ -module, define  ${}_f V = k \otimes_R V$ . Then  ${}_f V$  is naturally an  ${}_f H$ -module and specialization determines a functor from the category of  $H$ -modules to the category of  ${}_f H$ -modules. In the special case when  $R \subseteq k$  and  $\iota: R \rightarrow k$  is the inclusion, we replace the subscript  $\iota$  by  $k$ , so  ${}_k H = k \otimes_R H$  and  ${}_k V = k \otimes_R V$ . In addition, for  $h \in H$  and  $v \in V$ , we sometimes abuse notation slightly and denote  $1 \otimes h$  by  $h$  and  $1 \otimes v$  by  $v$ . With this convention,  ${}_k H$  is the space of all  $k$ -linear combinations of elements in  $H$ , and similarly for  ${}_k V$ .

There are two types of specializations of  $\mathbb{Z}[\mathbf{q}, \mathbf{a}]$  that relate  $\mathcal{H}_{b,n}$  to the representation theory of finite groups.

- (1) If  $f: \mathbb{Z}[\mathbf{q}, \mathbf{a}] \rightarrow k$  is a specialization such that  $f(\mathbf{q}) = 1$  and  $f(\mathbf{a}) = 0$ , then clearly

$${}_f \mathcal{H}_{b,n} \cong k[WD] \cong k[G(b, 1, n)].$$

- (2) If  $f: \mathbb{Z}[\mathbf{q}, \mathbf{a}] \rightarrow k$  is a specialization such that  $k$  is a field with characteristic zero,  $f(\mathbf{q}) = q$  is a prime power, and  $f(\mathbf{a}) = a$ , where  $q - 1 = ab$  with  $a$  and  $b$  relatively prime, then with the notation used in §1, it is shown in [1] that

$${}_f \mathcal{H}_{b,n} \cong \mathcal{H}_k(G, B_a),$$

where the algebra on the right-hand side is the convolution algebra of  $k$ -valued functions on  $G = \mathrm{GL}_n(\mathbb{F}_q)$  that are constant on  $(B_a, B_a)$ -double cosets. In this

isomorphism  $WD$  is identified with a group of monomial matrices in  $G$  and the basis element  $1 \otimes t_x$  of  ${}_f\mathcal{H}_{b,n}$  corresponds to  $|B_a|^{-1}$  times the characteristic function of the double coset  $B_axB_a$ .

In particular, if  $b = q - 1$  and  $a = 1$ , then  ${}_f\mathcal{H}_{q-1,n} \cong \mathcal{H}_k(G, U)$  and the presentation given by relations  $(\mathbf{r}_1) - (\mathbf{r}_6)$  is essentially that given by Yokonuma [13].

Taking  $k = \mathbb{C}$  in both cases above, it follows from the results in [1] and Tits' Deformation Theorem [5, 68.17] that  $\mathcal{H}_{\mathbb{C}}(G, B_a) \cong \mathbb{C}[G(b, 1, n)]$ , so as a special case,  $\mathcal{H}_{\mathbb{C}}(G, U) \cong \mathbb{C}[G(q - 1, 1, n)]$ .

**2.7. The ring  $A$  and the algebra  $\mathcal{H}$ .** Recall that  $n$  and  $b$  are positive integers,  $\mathbf{q}$  and  $\mathbf{a}$  are indeterminates, and  $\zeta$  is a complex primitive  $b^{\text{th}}$ -root of unity. In order to continue we need to replace the scalar ring  $\mathbb{Z}[\mathbf{q}, \mathbf{a}]$  by the smallest sufficiently large extension in which the constructions in this section and the next can be carried out.

Define  $A$  to be the smallest subring of an algebraic closure of  $\mathbb{C}(\mathbf{q}, \mathbf{a})$  with the following properties.

- (1)  $A$  contains  $1$ ,  $\mathbf{q}$ ,  $\mathbf{a}$ ,  $\zeta$ , and a square root of  $b^2\mathbf{a}^2 + 4\mathbf{q}$ , denoted by  $\sqrt{b^2\mathbf{a}^2 + 4\mathbf{q}}$ .
- (2)  $2$ ,  $b$ , and  $\mathbf{q}$  are units in  $A$ .
- (3) For  $k \in [n - 2]$  the elements  $1 + x + \cdots + x^k$  are units in  $A$ , where

$$x = \frac{b^2\mathbf{a}^2 + b\mathbf{a}\sqrt{b^2\mathbf{a}^2 + 4\mathbf{q}} + 2\mathbf{q}}{2\mathbf{q}}.$$

(It is shown in Lemma 2.8 that  $x$  is not a root of unity, so  $1 + x + \cdots + x^k \neq 0$  for all  $k > 0$ .)

More concisely, using the standard notation for localization and adjoining elements to a subring,

$$A = X^{-1} \left( Q^{-1} (Y^{-1} \mathbb{Z}[\mathbf{q}, \mathbf{a}]) [\sqrt{b^2\mathbf{a}^2 + 4\mathbf{q}}] \right),$$

where  $X = \{ \sum_{i=0}^k x^i \mid k \in [n - 2] \}$ ,  $Q = \{ \mathbf{q}^i \mid i \geq 0 \}$ , and  $Y = \{ 2^i b^j \mid i, j \geq 0 \}$ . By construction  $A$  is an integral domain. Let  $K$  denote the quotient field of  $A$ .

Define

$$\mathcal{H} = A \otimes_{\mathbb{Z}[\mathbf{q}, \mathbf{a}]} \mathcal{H}_{b,n} = {}_A\mathcal{H}_{b,n}.$$

In the rest of this paper we will be concerned with the  $A$ -algebra  $\mathcal{H}$  and its specializations.

**Lemma 2.8.** *The element  $x \in A$  is not a root of unity.*

*Proof.* Let  $I$  be the ideal in  $A$  generated by  $b\mathbf{a} - \mathbf{q} + 1$ . Then  $b\mathbf{a} + I = \mathbf{q} - 1 + I$  and so  $b^2\mathbf{a}^2 + 4\mathbf{q} + I = (\mathbf{q} + 1)^2 + I$ . It follows that  $\sqrt{b^2\mathbf{a}^2 + 4\mathbf{q}} + I = \pm(\mathbf{q} + 1) + I$ . Therefore,

$$x + I = \frac{b^2\mathbf{a}^2 + b\mathbf{a}\sqrt{b^2\mathbf{a}^2 + 4\mathbf{q}} + 2\mathbf{q}}{2\mathbf{q}} + I = \frac{\mathbf{q}^2 + 1 \pm (\mathbf{q}^2 - 1)}{2\mathbf{q}} + I = \mathbf{q}^{\pm 1} + I.$$

Now if  $x^k = 1$ , then  $\mathbf{q}^{\pm k} - 1 \in I \cap \mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}] = 0$ , and so  $k = 0$ . Thus,  $x$  is not a root of unity.  $\square$

**2.9. Characters, idempotents, and weights.** In this section we give the straightforward adaption of Thiem's constructions [12, Chapter 6] to the algebra  $\mathcal{H}$ .

Let  $X(D)$  denote the group of  $A$ -characters of  $D$ . That is, the group of all group homomorphisms  $\chi: D \rightarrow A^\times$ , where  $A^\times$  denotes the unit group of  $A$ . Note that  $|X(D)| = b^n$  because  $A$  contains a primitive  $b^{\text{th}}$  root of unity. An explicit construction of the characters in  $X(D)$  is given in 3.1.

For  $\chi \in X(D)$  define

$$e_\chi = b^{-n} \sum_{d \in D} \chi(d^{-1}) t_d \in \mathcal{D}.$$

Then  $e_\chi$  is the centrally primitive idempotent in  $\mathcal{D}$  with  $t_d e_\chi = \chi(d) e_\chi$  for  $d \in D$ . Moreover,  $\{e_\chi \mid \chi \in X(D)\}$  is an  $A$ -basis of  $\mathcal{D}$  and a complete set of orthogonal idempotents in  $\mathcal{D}$ , and hence in  $\mathcal{H}$ .

The group  $W$  normalizes  $D$  and so acts on  $X(D)$  with  $(w \cdot \chi)(d) = \chi(w^{-1}dw)$  for  $w \in W$ ,  $\chi \in X(D)$ , and  $d \in D$ . Let  $\mathcal{C}$  denote the set of orbits of  $W$  in  $X(D)$ , and for each orbit  $\alpha \in \mathcal{C}$  fix a representative  $\chi_\alpha \in \alpha$ .

Using Theorem 2.5 it is straightforward to check that  $\{t_w e_\chi \mid w \in W, \chi \in X(D)\}$  is an  $A$ -basis of  $\mathcal{H}$  and that

$$(a) \quad t_w e_\chi = e_{w \cdot \chi} t_w$$

for  $w \in W$  and  $\chi \in X(D)$ .

Suppose  $\alpha \in \mathcal{C}$ . Define

$$\mathcal{H}_\alpha = e_{\chi_\alpha} \mathcal{H} e_{\chi_\alpha} \quad \text{and} \quad e_\alpha = \sum_{\chi \in \alpha} e_\chi.$$

Then  $\mathcal{H}_\alpha$  is a subalgebra of  $\mathcal{H}$  and  $e_\alpha$  is an idempotent in  $\mathcal{D}$ .

It is shown in [1] that  $t_w$  is a unit in  $\mathcal{H}$  for each  $w$  in  $W$ . Thus, by (a),

$$\mathcal{H} e_\chi = \mathcal{H} t_w e_\chi = \mathcal{H} e_{w \cdot \chi} t_w,$$

and so right multiplication by  $t_w$  defines an  $\mathcal{H}$ -module isomorphism  $\mathcal{H} e_\chi \cong \mathcal{H} e_{w \cdot \chi}$ . This implies that  $e_\chi \mathcal{H} e_\chi \cong e_{w \cdot \chi} \mathcal{H} e_{w \cdot \chi}$  for all  $w \in W$ . In particular, up to isomorphism,  $\mathcal{H}_\alpha$  depends only on  $\alpha$  and not the choice of  $\chi_\alpha$ .

It follows from (a) that  $e_\alpha$  is in the center of  $\mathcal{H}$ . Therefore, the set  $\{e_\alpha \mid \alpha \in \mathcal{C}\}$  is a complete set of orthogonal, central idempotents in  $\mathcal{H}$ , and there are direct sum decompositions

$$\mathcal{H} e_\alpha \cong \bigoplus_{\chi \in \alpha} \mathcal{H} e_\chi \quad \text{and} \quad \mathcal{H} \cong \bigoplus_{\alpha \in \mathcal{C}} \mathcal{H} e_\alpha \cong \bigoplus_{\alpha \in \mathcal{C}} \left( \bigoplus_{\chi \in \alpha} \mathcal{H} e_\chi \right),$$

where each  $\mathcal{H} e_\alpha$  is a two-sided ideal.

Now suppose  $V$  is an  $\mathcal{H}$ -module. For  $\chi \in X(D)$  define

$$V_\chi = e_\chi V = \{v \in V \mid t_d v = \chi(d) v \ \forall d \in D\}.$$

Then  $V_\chi$  is an  $A$ -submodule of  $V$  and

$$V \cong \bigoplus_{\chi \in X(D)} V_\chi$$

is a decomposition of  $V$  as a direct sum of  $A$ -modules. We can think of characters in  $X(D)$  as weights and then  $V_\chi$  is the  $\chi$ -weight space of  $V$ .

If  $w \in W$ ,  $d \in D$ , and  $v \in V_\chi$ , then by Theorem 2.5,

$$(b) \quad t_d(t_w v) = t_w t_{w^{-1}dw} v = \chi(w^{-1}dw) t_w v = (w \cdot \chi)(d) t_w v,$$

and so multiplication by  $t_w$  defines an  $A$ -linear isomorphism  $V_\chi \cong V_{w \cdot \chi}$ . It follows that

$$\sum_{w \in W} t_w V_\chi = \sum_{w \in W} V_{w \cdot \chi}$$

is an  $\mathcal{H}$ -submodule of  $V$ .

For  $\alpha \in \mathcal{C}$  define

$$V_\alpha = e_\alpha V = \sum_{\chi \in \alpha} V_\chi.$$

Then  $V_\alpha$  is an  $\mathcal{H}$ -submodule of  $V$  and

$$(c) \quad V \cong \bigoplus_{\alpha \in \mathcal{C}} V_\alpha$$

is a decomposition of  $V$  as a direct sum of  $\mathcal{H}$ -submodules.

**2.10. Blocks.** For a ring  $R$ , let  $R\text{-mod}$  denote the category of left  $R$ -modules.

Suppose  $\alpha \in \mathcal{C}$ . Define  $\mathcal{O}_\alpha$  to be the full subcategory of  $\mathcal{H}\text{-mod}$  with objects the collection of all  $\mathcal{H}$ -modules  $V$  such that  $V = V_\alpha$ . At the risk of abusing terminology we say that  $\mathcal{O}_\alpha$  is a block of  $\mathcal{H}$ .

If  $V$  and  $V'$  are  $\mathcal{H}$ -modules and  $\psi: V \rightarrow V'$  is an  $\mathcal{H}$ -module homomorphism, then  $\psi(V_\alpha) \subseteq V'_\alpha$  for all  $\alpha \in \mathcal{C}$ . Thus, if  $\alpha, \beta \in \mathcal{C}$  with  $\alpha \neq \beta$ ,  $V$  is a module in  $\mathcal{O}_\alpha$ , and  $V'$  is a module in  $\mathcal{O}_\beta$ , then  $\text{Hom}_{\mathcal{H}}(V, V') = 0$ . Hence, it follows from the decompositions in 2.9(c) that

$$(a) \quad \mathcal{H}\text{-mod} \simeq \bigoplus_{\alpha \in \mathcal{C}} \mathcal{O}_\alpha$$

is a decomposition of  $\mathcal{H}\text{-mod}$  as a direct sum of abelian subcategories.

If  $V$  is an  $\mathcal{H}$ -module, then  $V_{\chi_\alpha} = e_{\chi_\alpha} V$  is an  $\mathcal{H}_\alpha$ -module. Let

$$G_\alpha: \mathcal{O}_\alpha \rightarrow \mathcal{H}_\alpha\text{-mod}$$

be the restriction functor defined on objects by  $G_\alpha(V) = V_{\chi_\alpha}$  and on homomorphisms by restriction.

If  $V'$  is an  $\mathcal{H}_\alpha$ -module, then  $e_{\chi_\alpha}(\mathcal{H}e_{\chi_\alpha} \otimes_{\mathcal{H}_\alpha} V') = \mathcal{H}_\alpha \otimes_{\mathcal{H}_\alpha} V' \cong V'$  and so the  $\mathcal{H}$ -module  $\mathcal{H}e_{\chi_\alpha} \otimes_{\mathcal{H}_\alpha} V'$  is in  $\mathcal{O}_\alpha$ . Let

$$F_\alpha: \mathcal{H}_\alpha\text{-mod} \rightarrow \mathcal{O}_\alpha$$



be the induction functor defined on objects by  $F_\alpha(V') = \mathcal{H} \otimes_{\mathcal{H}_\alpha} V' = \mathcal{H}e_{\chi_\alpha} \otimes_{\mathcal{H}_\alpha} V'$  and on morphisms by  $F_\alpha(\psi) = \text{id} \otimes \psi$ .

Suppose  $f: A \rightarrow k$  is a specialization. Then  $f(b)$  is a unit in  $k$  and it follows that  $f(\zeta)$  is a primitive  $b^{\text{th}}$  root of unity. (If  $f(\zeta)$  is an  $l^{\text{th}}$  root of unity and  $b = lm$ , then  $1 + f(\zeta)^l + \cdots + (f(\zeta)^l)^{m-1} = m$  is not equal to zero in  $k$ . But  $\zeta^l$  is an  $m^{\text{th}}$  root of unity, so  $1 + f(\zeta^l) + \cdots + f(\zeta^l)^{m-1} = 0$ , which is a contradiction unless  $m = 1$  and  $l = b$ .) Therefore, the rule  $\chi \mapsto f \circ \chi$  defines an isomorphism between  $X(D)$  and  $X_k(D)$  so the preceding constructions all apply in  ${}_f\mathcal{H}\text{-mod}$  and there are categories  ${}_f\mathcal{O}$  and functors  ${}_fF_\alpha$  and  ${}_fG_\alpha$ .

The next theorem is an analog of results of Lusztig [11, §34] and Jacon and Poulain d'Andecy [8, §3].

**Theorem 2.11.** *Suppose  $\alpha \in \mathcal{C}$  and  $f: A \rightarrow k$  is a specialization. Then the pair of functors  $({}_fF_\alpha, {}_fG_\alpha)$  is an adjoint equivalence of categories. In particular, the block  ${}_f\mathcal{O}_\alpha$  of  ${}_f\mathcal{H}$  is naturally equivalent to the category of  ${}_f\mathcal{H}_\alpha$ -modules.*

*Proof.* To help minimize subscripts, in this proof we suppress the specialization from the notation. For example we denote  ${}_fF_\alpha$ ,  ${}_f\mathcal{H}$ , and  $1 \otimes e_{\chi_\alpha}$ , simply by  $F_\alpha$ ,  $\mathcal{H}$ , and  $e_{\chi_\alpha}$ , respectively.

It is well-known and straightforward to check that  $(F_\alpha, G_\alpha)$  is an adjoint pair. As observed above, if  $V'$  is an  $\mathcal{H}_\alpha$ -module, then  $G_\alpha(F_\alpha(V')) = e_{\chi_\alpha}(\mathcal{H}e_{\chi_\alpha} \otimes_{\mathcal{H}_\alpha} V')$  is naturally isomorphic to  $V'$  and so  $G_\alpha F_\alpha$  is naturally equivalent to the identity functor. To complete the proof it is enough to show that if  $V$  is an  $\mathcal{H}$ -module in  $\mathcal{O}_\alpha$  then the natural map

$$\gamma: \mathcal{H}e_{\chi_\alpha} \otimes_{\mathcal{H}_\alpha} V_{\chi_\alpha} \rightarrow V \quad \text{with} \quad \gamma(h \otimes v) = hv$$

is an  $\mathcal{H}$ -module isomorphism. The map  $\gamma$  is obviously  $\mathcal{H}$ -linear and so it is enough to show that it is an  $A$ -module isomorphism.

Suppose  $w \in W$ ,  $\chi \in X(D)$  with  $\chi \neq \chi_\alpha$ , and  $v \in V_{\chi_\alpha}$ . Then

$$t_w e_\chi \otimes v = t_w e_\chi \otimes e_{\chi_\alpha} v = t_w e_\chi e_{\chi_\alpha} \otimes v = 0.$$

Because the set  $\{t_w e_\chi \mid w \in W, \chi \in X(D)\}$  is an  $A$ -basis of  $\mathcal{H}$  it follows that the set  $\{t_w \otimes v \mid w \in W, v \in V_{\chi_\alpha}\}$  spans  $F_\alpha G_\alpha(V)$ . Using Theorem 2.5 we have

$$t_d(t_w \otimes v) = t_w t_{w^{-1}d} \otimes v = t_w t_{w^{-1}d} e_{\chi_\alpha} \otimes v = (w \cdot \chi_\alpha)(d)(t_w \otimes v),$$

and so  $F_\alpha G_\alpha(V)_{w \cdot \chi_\alpha} = t_w \otimes V_{\chi_\alpha}$ . Moreover, if  $w \cdot \chi_\alpha = \chi_\alpha$ , then  $t_w e_{\chi_\alpha} = e_{\chi_\alpha} t_w e_{\chi_\alpha} \in \mathcal{H}_\alpha$  by 2.9(a), and so  $t_w \otimes v = 1 \otimes t_w e_{\chi_\alpha} v \in 1 \otimes V_{\chi_\alpha}$ . It follows that

$$F_\alpha G_\alpha(V)_{\chi_\alpha} = \sum_{w \cdot \chi_\alpha = \chi_\alpha} t_w \otimes V_{\chi_\alpha} = 1 \otimes V_{\chi_\alpha}.$$

The restriction of  $\gamma$  to  $1 \otimes V_{\chi_\alpha}$  is obviously an isomorphism onto  $V_{\chi_\alpha}$ . Because  $F_\alpha G_\alpha(V)$  and  $V$  are in  $\mathcal{O}_\alpha$  and left multiplication by the elements  $t_w$  for  $w \in W$  transitively permutes the non-zero weight spaces, the multiplication map  $\gamma$  carries  $F_\alpha G_\alpha(V)_\chi$  isomorphically to  $V_\chi$  for all  $\chi \in X(D)$ . This implies that  $\gamma$  is an isomorphism.  $\square$

It is not hard to see that  $\mathcal{O}_\alpha$  is naturally isomorphic to the category of  $\mathcal{H}e_\alpha$ -modules and so it follows from the theorem that  $\mathcal{H}e_\alpha$  and  $\mathcal{H}_\alpha$  are Morita equivalent. Jacon and Poulain d'Andecy prove a more explicit result in [8]. Using the choice of orbit representatives in 3.1, their arguments can be easily adapted to construct an explicit isomorphism  $\mathcal{H}e_\alpha \cong M_{r_\alpha}(\mathcal{H}_\alpha)$ , where  $r_\alpha = |\alpha|$ .

### 3. CONSTRUCTION OF $\mathcal{H}$ -MODULES

The block decomposition in 2.10(a) and Theorem 2.11 reduce the study of  $\mathcal{H}$ -modules to the study of  $\mathcal{H}_\alpha$ -modules. It is shown below that  $\mathcal{H}_\alpha$  is isomorphic to a tensor product of Iwahori-Hecke algebras of type  $A$ . Irreducible representations for Iwahori-Hecke algebras of type  $A$  have been constructed by various authors. In this paper we use Hoefsmit's construction to define a family of  $\mathcal{H}_\alpha$ -modules. Inducing these modules to  $\mathcal{H}$  we obtain a family of  $\mathcal{H}$ -modules that gives rise to a complete set of irreducible  ${}_f\mathcal{H}$ -modules, for any specialization  $f: A \rightarrow k$  with the property that  ${}_f\mathcal{H}_\alpha$  is semisimple for all  $\alpha$ . A particular case is the inclusion of  $A$  in its quotient field  $K$ . In this case,  ${}_K\mathcal{H}$  is a split semisimple  $K$  algebra and we obtain a complete set of irreducible  ${}_K\mathcal{H}$ -modules, each of which is absolutely irreducible.

This section is organized as follows. First, the combinatorial constructions we use are collected in 3.1–3.7. This is followed in 3.10 by a review of Hoefsmit's construction in the form used later. The structure of  $\mathcal{H}_\alpha$  is described and the  $\mathcal{H}_\alpha$ -modules we need are constructed in 3.11–3.16. With these  $\mathcal{H}_\alpha$ -modules in hand, we can state and prove Theorem 3.17, the main result in this section. Finally, consequences for semisimple specializations of  $\mathcal{H}$  are given in Corollary 3.19 and Corollary 3.20.

**3.1. Pseudo-compositions and orbit representatives.** Because  $D \cong \mu_b^n$ , we have  $X(D) \cong X(\mu_b^n) \cong X(\mu_b)^n$  and so we may use  $[b]^n$  to index the characters of  $D$ . We make this identification precise and choose the usual orbit representatives for the  $W$  action as follows.

For  $(i_1, \dots, i_n) \in [b]^n$  define

$$\chi_{(i_1, \dots, i_n)}: D \rightarrow A \quad \text{by} \quad \chi_{(i_1, \dots, i_n)} \left( \begin{bmatrix} \zeta^{p_1} & & \\ & \ddots & \\ & & \zeta^{p_n} \end{bmatrix} \right) = \zeta^{i_1 p_1 + \dots + i_n p_n}.$$

Then  $X(D) = \{ \chi_{\bar{\tau}} \mid \bar{\tau} \in [b]^n \}$ . For example,  $\chi_{(1, \dots, 1)}$  is the determinant character and  $\chi_{(b, \dots, b)}$  is the trivial character. Notice that

$$(a) \quad T_1^{p_1} \dots T_n^{p_n} e_{\chi_{\bar{\tau}}} = \zeta^{i_1 p_1 + \dots + i_n p_n} e_{\chi_{\bar{\tau}}}.$$

When considered as a permutation group,  $W$  acts on  $[b]^n$  on the right by place permutation: for  $w \in W$  and  $\bar{\tau} = (i_1, \dots, i_n) \in [b]^n$ ,  $\bar{\tau} \cdot w = (i_{w(1)}, \dots, i_{w(n)})$ . The proof of the next lemma is straightforward and is omitted.

**Lemma 3.2.** *Suppose  $w \in W$  and  $\bar{\tau} \in [b]^n$ . Then*

$$w \cdot \chi_{\bar{\tau}} = \chi_{\bar{\tau} \cdot w^{-1}}.$$

It follows from the lemma that  $\chi_{\bar{\tau}}$  and  $\chi_{\bar{j}}$  lie in the same  $W$ -orbit if and only if  $\bar{j}$  can be obtained from  $\bar{\tau}$  by permuting the entries.

3.3. A pseudo-composition of  $n$  with  $b$  parts, sometimes called a  $b$ -composition of  $n$ , is a  $b$ -tuple  $(m_1, \dots, m_b)$  of non-negative integers such that  $m_1 + \dots + m_b = n$ . Let  $\mathcal{C}(n, b)$  denote the set of pseudo-compositions of  $n$  with  $b$  parts.

Define

$$\pi: [b]^n \rightarrow \mathcal{C}(n, b) \quad \text{by} \quad \pi(i_1, \dots, i_n) = (m_1, \dots, m_b),$$

where for  $j \in [b]$ ,  $m_j = |\{l \mid i_l = j\}|$  is the multiplicity of  $j$  in the tuple  $(i_1, \dots, i_n)$ . Then  $\pi$  is an orbit map for the action of  $W$  on  $[b]^n$  and  $\chi_{\bar{\tau}}$  and  $\chi_{\bar{j}}$  lie in the same  $W$ -orbit if and only if  $\pi(\bar{\tau}) = \pi(\bar{j})$ . Abusing notation slightly, define

$$\pi: X(D) \rightarrow \mathcal{C}(n, b) \quad \text{by} \quad \pi(\chi_{\bar{\tau}}) = \pi(\bar{\tau}).$$

Then,  $\chi$  and  $\chi'$  lie in the same  $W$ -orbit if and only if  $\pi(\chi) = \pi(\chi')$ . From now on we use the map  $\pi$  to identify the set  $\mathcal{C}$  of  $W$ -orbits in  $X(D)$  with the set  $\mathcal{C}(n, b)$ .

For  $\alpha = (m_1, \dots, m_b) \in \mathcal{C}(n, b)$  define

$$\bar{\tau}_\alpha = (\underbrace{1, \dots, 1}_{m_1}, \dots, \underbrace{b, \dots, b}_{m_b}) \in [b]^n \quad \text{and} \quad \chi_\alpha = \chi_{\bar{\tau}_\alpha} \in X(D).$$

For example, if  $\alpha = (0, 2, 3, 2) \in \mathcal{C}(7, 4)$ , then  $\bar{\tau}_\alpha = (2, 2, 3, 3, 3, 4, 4) \in [4]^7$ , and if  $\beta = (0, 0, 0, 7) \in \mathcal{C}(7, 4)$ , then  $\chi_\beta$  is the trivial character. It is easy to see that  $\{\bar{\tau}_\alpha \mid \alpha \in \mathcal{C}(n, b)\}$  is the set of all non-decreasing tuples in  $[b]^n$  and is a complete set of orbit representatives for the action of  $W$  on  $[b]^n$ . Thus,  $\{\chi_\alpha \mid \alpha \in \mathcal{C}(n, b)\}$  is a complete set of orbit representatives in  $X(D)$ .

3.4. **Multipartitions and tableaux.** A partition of  $n$  is a tuple  $\lambda = (n_1, \dots, n_p)$  of positive integers such that  $n_1 \geq \dots \geq n_p$  and  $n = n_1 + \dots + n_p$ . The integers  $n_i$  are the parts of  $\lambda$  and  $|\lambda| = n$  is the size of  $\lambda$ . By definition, the empty tuple is a partition of 0 called the empty partition and denoted by  $\emptyset$ . We have  $|\emptyset| = 0$ . A partition is a partition of a non-negative integer.

A  $b$ -partition of  $n$ , sometimes called a  $b$ -multipartition of  $n$ , is a  $b$ -tuple  $\lambda = (\lambda^1, \dots, \lambda^b)$  of partitions, some of which could be the empty partition, such that  $|\lambda^1| + \dots + |\lambda^b| = n$ . Let  $b\mathcal{P}(n)$  denote the set of  $b$ -partitions of  $n$ .

For  $\lambda = (\lambda^1, \dots, \lambda^b)$ , a  $b$ -partition of  $n$ , define

$$|\lambda| = (|\lambda^1|, \dots, |\lambda^b|).$$

Then clearly  $|\lambda| \in \mathcal{C}(n, b)$ . Note that a  $b$ -partition of  $n$  is given by a pseudo-composition of  $n$  with  $b$  parts, say  $\alpha = (m_1, \dots, m_b)$ , together with a partition of  $m_i$  for each  $i \in [b]$ .

A partition of  $n$  may be visualized as a Young diagram and we frequently identify a partition with its corresponding Young diagram without comment. Suppose  $\lambda = (\lambda^1, \dots, \lambda^b) \in b\mathcal{P}(n)$ . The Young diagram of  $\lambda$  is the  $b$ -tuple of Young diagrams whose  $i^{\text{th}}$  entry is the Young diagram of  $\lambda^i$ .

A Young  $b$ -tableau with shape  $\lambda$  is a bijection between  $\{1, \dots, n\}$  and the boxes in the Young diagram of  $\lambda$ . Young  $b$ -tableaux with shape  $\lambda$  are usually visualized by filling in the boxes in the Young diagram of  $\lambda$  with the integers  $1, \dots, n$ . Let  $YT^\lambda$  denote the set of Young  $b$ -tableaux with shape  $\lambda$ . If  $\tau = (\tau^1, \dots, \tau^b)$  is a Young  $b$ -tableau with shape  $\lambda$ , then  $\tau$  is standard if the entries in each row and column of  $\tau^i$  are increasing for each  $i \in [b]$ . Let  $SYT^\lambda$  denote the set of standard Young  $b$ -tableaux with shape  $\lambda$ .

For example, if

$$\lambda = ((2, 1), \emptyset, (3, 2), (1, 1)) = \left( \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}, \emptyset, \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|} \hline \\ \hline \end{array} \right) \in 4\mathcal{P}(10),$$

then

$$|\lambda| = (3, 0, 5, 2) \in \mathcal{C}(10, 4) \quad \text{and} \quad \tau = \left( \begin{array}{|c|c|} \hline 3 & 7 \\ \hline 9 & \\ \hline \end{array}, \emptyset, \begin{array}{|c|c|c|} \hline 2 & 5 & 10 \\ \hline 4 & 8 & \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 6 \\ \hline \end{array} \right) \in SYT^\lambda.$$

Suppose  $\lambda = (\lambda^1, \dots, \lambda^b) \in b\mathcal{P}(n)$  and  $\tau = (\tau^1, \dots, \tau^b) \in YT^\lambda$ . For  $i \in [n]$ , define  $\tau_i = j$  if  $i$  appears in  $\tau^j$ . In other words,  $\tau_i = j$  if the box containing  $i$  in the Young diagram of  $\lambda$  is in  $\lambda^j$ . By construction, the tuple  $(\tau_1, \dots, \tau_n)$  is in  $[b]^n$ . Define

$$\xi: YT \rightarrow [b]^n \quad \text{by} \quad \xi(\tau) = (\tau_1, \dots, \tau_n).$$

Continuing the example above we have

$$\xi \left( \left( \begin{array}{|c|c|} \hline 3 & 7 \\ \hline 9 & \\ \hline \end{array}, \emptyset, \begin{array}{|c|c|c|} \hline 2 & 5 & 10 \\ \hline 4 & 8 & \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 6 \\ \hline \end{array} \right) \right) = (4, 3, 1, 3, 3, 4, 1, 3, 1, 3) \in [4]^{10}.$$

Next, define

$$YT_0^\lambda = \{ \tau \in YT^\lambda \mid \tau_1 \leq \dots \leq \tau_n \} \quad \text{and} \quad SYT_0^\lambda = YT_0^\lambda \cap SYT.$$

Thus, if  $|\lambda| = (m_1, \dots, m_b)$ , then  $\tau = (\tau^1, \dots, \tau^b) \in YT_0^\lambda$  if and only if

$$\tau_1 = \dots = \tau_{m_1} = 1, \quad \tau_{m_1+1} = \dots = \tau_{m_1+m_2} = 2, \quad \text{and so on.}$$

For example,

$$\left( \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}, \emptyset, \begin{array}{|c|c|c|} \hline 4 & 6 & 7 \\ \hline 5 & 8 & \\ \hline \end{array}, \begin{array}{|c|} \hline 9 \\ \hline 10 \\ \hline \end{array} \right) \in SYT_0^\lambda,$$

where as above  $\lambda = ((2, 1), \emptyset, (3, 2), (1, 1))$ .

As a matter of convention, we identify 1-partitions with partitions and 1-tableaux with tableaux. For example,  $\mathcal{P}(m) = 1\mathcal{P}(m)$  is the set of partitions of  $m$ , and if  $\hat{\lambda} \in \mathcal{P}(m)$ , then  $SYT^{\hat{\lambda}}$  denotes the set of standard Young tableaux with shape  $\hat{\lambda}$ .

3.5. Suppose  $\lambda \in b\mathcal{P}(n)$ . The group  $W$  acts on  $YT^\lambda$  in the obvious way: for  $w \in W$  and  $\tau \in YT^\lambda$ ,  $w \cdot \tau$  is the Young  $b$ -tableau obtained by applying the permutation  $w$  to the entries of  $\tau$ . Define

$$\chi_\tau = \chi_{\xi(\tau)} \in X(D).$$

**Lemma 3.6.** *Suppose  $\tau \in YT^\lambda$ . Then  $w \cdot \chi_\tau = \chi_{w \cdot \tau}$ .*

*Proof.* Note that the box that contains  $w^{-1}(j)$  in  $\tau$  is the same as the box that contains  $j$  in  $w \cdot \tau$ , so  $\tau_{w^{-1}(j)} = (w \cdot \tau)_j$ . Therefore

$$w \cdot \chi_\tau = w \cdot \chi_{(\tau_1, \dots, \tau_n)} = \chi_{(\tau_{w^{-1}(1)}, \dots, \tau_{w^{-1}(n)})} = \chi_{w \cdot \tau}$$

by Lemma 3.2.  $\square$

3.7. To summarize, given  $\lambda \in b\mathcal{P}(n)$ ,  $\tau \in YT^\lambda$ , and  $\alpha \in \mathcal{C}(n, b)$ , we have characters

$$\chi_\tau = \chi_{\xi(\tau)}, \quad \chi_\alpha = \chi_{\bar{\tau}_\alpha}, \quad \text{and} \quad \chi_{|\lambda|} \text{ in } X(D).$$

It is easy to see that  $\pi(\xi(\tau)) = |\lambda|$  and so  $\chi_\tau$  is in the  $W$ -orbit of  $\chi_{|\lambda|}$ . In addition, if  $\tau_0 \in YT_0^\lambda$  and  $|\lambda| = \alpha$ , then  $\xi(\tau_0) = \bar{\tau}_\alpha$  and

$$(a) \quad \chi_{\tau_0} = \chi_{|\lambda|} = \chi_\alpha.$$

3.8. **A factorization of b-tableaux.** Now suppose  $\alpha = (m_1, \dots, m_b) \in \mathcal{C}(n, b)$ . Define  $\bar{m}_0 = 0$ , and for  $j \in [b]$  define  $\bar{m}_j = m_1 + \dots + m_j$  and  $M_j = [\bar{m}_j] \setminus [\bar{m}_{j-1}]$ . Define the Young subgroup  $W_\alpha$  of  $W$  by

$$W_\alpha = \{ w \in W \mid \forall j \in [b], w(M_j) = M_j \}.$$

It is easy to see that  $W_\alpha$  is the stabilizer of  $\chi_\alpha$  in  $W$  and that  $W_\alpha \cong \Sigma_{m_1} \times \dots \times \Sigma_{m_b}$ , where  $\Sigma_0$  is understood to be the trivial group. Let  $W^\alpha$  denote the set of minimal length left coset representatives of  $W_\alpha$  in  $W$ . Considering  $w \in W$  as a permutation, that is, as a bijection  $w: [n] \rightarrow [n]$ , one may consider the restriction of  $w$  to each  $M_j$  as a function  $w|_{M_j}: M_j \rightarrow [n]$ , and then

$$W^\alpha = \{ w \in W \mid \forall j \in [b], w|_{M_j} \text{ is increasing} \}.$$

It is well-known that the multiplication map  $W^\alpha \times W_\alpha \rightarrow W$  is a bijection.

Suppose  $\lambda = (\lambda^1, \dots, \lambda^b) \in b\mathcal{P}(n)$ . Then  $|\lambda| \in \mathcal{C}(n, b)$  and so  $W_{|\lambda|}$  and  $W^{|\lambda|}$  are defined. Let  $\varphi: W^{|\lambda|} \times SYT_0^\lambda \rightarrow YT^\lambda$  be the action map defined by  $\varphi(w, \tau_0) = w \cdot \tau_0$ .

**Proposition 3.9.** *The mapping  $\varphi: W^{|\lambda|} \times SYT_0^\lambda \rightarrow YT^\lambda$  is injective with image equal to  $SYT^\lambda$  and so the  $W$ -action on  $YT^\lambda$  induces a bijection between  $W^{|\lambda|} \times SYT_0^\lambda$  and  $SYT^\lambda$ . Therefore, given  $\tau \in SYT^\lambda$ , there is a unique  $w \in W^{|\lambda|}$  and a unique  $\tau_0 \in SYT_0^\lambda$  such that  $\tau = w \cdot \tau_0$ .*

*Proof.* Suppose  $|\lambda| = (m_1, \dots, m_b)$  and that for  $j \in [b]$ ,  $\bar{m}_j$  and  $M_j$  are defined as in 3.8. Notice that  $\tau = (\tau^1, \dots, \tau^b) \in YT_0^\lambda$  if and only if for all  $j \in [b]$ ,  $M_j$  is the set of entries of  $\tau^j$ .

To show that  $\varphi$  is injective, suppose  $w, w' \in W^{|\lambda|}$ ,  $\tau_0, \tau'_0 \in SYT_0^\lambda$ , and  $w \cdot \tau_0 = w' \cdot \tau'_0$ . Then  $w^{-1}w' \cdot \tau'_0 = \tau_0$ , so  $w^{-1}w'(M_j) = M_j$  for all  $j \in [b]$ . But then  $w^{-1}w' \in W_{|\lambda|}$ , so  $w = w'$ , which implies that  $\tau_0 = \tau'_0$  as well.

Because  $w \in W^{|\lambda|}$  if and only if the restriction of  $w$  to  $M_j$  is an increasing function (when  $w$  is considered as a permutation), and a standard Young  $b$ -tableau  $\tau_0 = (\tau_0^1, \dots, \tau_0^b)$  with shape  $\lambda$  is in  $SYT_0^\lambda$  if and only if  $M_j$  is the set of entries of  $\tau_0^j$  for  $j \in [b]$ , it is clear that if  $w \in W^{|\lambda|}$  and  $\tau_0 \in SYT_0^\lambda$ , then  $w \cdot \tau_0$  is standard. Thus, the image of  $\varphi$  is contained in  $SYT^\lambda$ .

To show that the image of  $\varphi$  is equal to  $SYT^\lambda$ , suppose  $\tau = (\tau^1, \dots, \tau^b) \in SYT^\lambda$ . For  $j \in [b]$  let  $N_j = \{i_1^j, \dots, i_{m_j}^j\}$  be the set of entries of  $\tau^j$ , where  $i_1^j < \dots < i_{m_j}^j$ . Define a permutation  $w \in W$  by

$$w(\overline{m}_{j-1} + l) = i_l^j \quad \text{for } j \in [b] \quad \text{and } l \in [m_j].$$

Then  $w \in W^{|\lambda|}$ . Moreover,  $w^{-1}(N_j) = M_j$  and the restriction of  $w^{-1}$  to  $N_j$  is increasing for all  $j \in [b]$ . This implies that  $w^{-1} \cdot \tau \in YT_0^\lambda$  and  $w^{-1} \cdot \tau$  is standard, so  $w^{-1} \cdot \tau \in SYT_0^\lambda$ . Clearly  $\varphi(w, w^{-1} \cdot \tau) = \tau$  and hence the image of  $\varphi$  is equal to  $SYT^\lambda$ .  $\square$

**3.10. Hoefsmit's modules for Iwahori-Hecke algebras of type A.** Suppose  $m > 1$  is an integer. The Iwahori-Hecke algebra of type  $A_{m-1}$  over  $A$  with parameters  $\mathbf{q}$  and  $b\mathbf{a}$  is the unital  $A$ -algebra  $\mathcal{I}^m$  with generators

$$S_1, \dots, S_{m-1}$$

and relations

$$\begin{aligned} S_i S_j &= S_j S_i && \text{for } i, j \in [m-1] \text{ with } |i-j| > 1, \\ S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1} && \text{for } i \in [m-2], \text{ and} \\ S_i^2 &= \mathbf{q} + b\mathbf{a} S_i && \text{for } i \in [m-1]. \end{aligned}$$

It is shown in [2, §IV.2 Exercise 24] that  $\dim \mathcal{I}^m = m!$  and that  $\mathcal{I}^m$  has an  $A$ -basis  $\{S_\sigma \mid \sigma \in \Sigma_m\}$  such that if  $s_j$  is the transposition  $(j \ j+1)$  in  $\Sigma_m$ , then  $S_{s_j} = S_j$  and

$$S_{s_j} S_\sigma = \begin{cases} S_{s_j \sigma} & \text{if } \ell(s_j \sigma) = \ell(\sigma) + 1 \\ \mathbf{q} S_{s_j \sigma} + b\mathbf{a} S_\sigma & \text{if } \ell(s_j \sigma) = \ell(\sigma) - 1. \end{cases}$$

Define  $\mathcal{I}^0 = \mathcal{I}^1 = A$ .

Recall the element  $x \in A$  defined in 2.7 and that  $\sqrt{b\mathbf{a}^2 + 4\mathbf{q}}$  is a fixed square root of  $b^2\mathbf{a}^2 + 4\mathbf{q}$ . Define

$$y = \frac{b\mathbf{a} + \sqrt{b^2\mathbf{a}^2 + 4\mathbf{q}}}{2\mathbf{q}}.$$

Then  $x = \mathbf{q}y^2 = b\mathbf{a}y + 1$  and  $y$  is a unit in  $A$  with  $y^{-1} = (b\mathbf{a} - \sqrt{b^2\mathbf{a}^2 + 4\mathbf{q}})/2$ . Notice that if  $\widehat{S}_l \in \mathcal{I}^m$  is defined by  $\widehat{S}_l = yS_l$ , then  $\widehat{S}_l^2 = x1 + (x-1)\widehat{S}_l$ .

If  $\hat{\tau}$  is a Young 1-tableau and  $i$  and  $j$  are entries in  $\hat{\tau}$ , then the axial distance from  $i$  to  $j$  in  $\hat{\tau}$  is

$$\delta_\tau(i, j) = (c_j - r_j) - (c_i - r_i),$$

when the box in  $\hat{\tau}$  that contains  $i$  is in row  $r_i$  and column  $c_i$ , and similarly for  $j$ . For example, if  $\tau = \begin{array}{|c|c|c|} \hline 2 & 5 & 10 \\ \hline 4 & 8 & \\ \hline \end{array}$ , then  $\delta_\tau(4, 10) = (3-1) - (1-2) = 3 = -\delta_\tau(10, 4)$ .

For a partition  $\hat{\lambda}$  of  $m$ , let  $P^{\hat{\lambda}}$  be a free  $A$ -module with basis indexed by  $SYT^{\hat{\lambda}}$ , the set of standard Young tableaux with shape  $\hat{\lambda}$ . Say  $\{p_{\hat{\tau}} \mid \hat{\tau} \in SYT^{\hat{\lambda}}\}$  is an  $A$ -basis of

$P^\lambda$ . Hoefsmit [7] has shown that there is an  $\mathcal{I}^m$ -module structure on  $P^\lambda$  such that for  $l \in [m-1]$  and  $\hat{\tau} \in SYT^\lambda$ ,

$$(a) \quad S_l \cdot p_{\hat{\tau}} = \frac{x-1}{y(1-x^k)} p_{\hat{\tau}} + \frac{x-x^k}{y(1-x^k)} p_{s_l \cdot \hat{\tau}},$$

where  $k = \delta_{\hat{\tau}}(l+1, l)$  and  $p_{s_l \cdot \hat{\tau}}$  is taken to be zero if  $s_l \cdot \hat{\tau}$  is not standard.

**3.11. The algebra  $\mathcal{H}_\alpha$  and  $\mathcal{H}_\alpha$ -modules.** We now take up the structure of the algebra  $\mathcal{H}_\alpha$  and the construction of  $\mathcal{H}_\alpha$ -modules. From this subsection until the statement of Theorem 3.17,  $\alpha = (m_1, \dots, m_b) \in \mathcal{C}(n, b)$  denotes a fixed pseudo-composition of  $n$  with  $b$  parts.

**Lemma 3.12.** *Consider the subalgebra  $\mathcal{H}_\alpha$  of  $\mathcal{H}$  and the right  $\mathcal{H}_\alpha$ -module  $\mathcal{H}e_{\chi_\alpha}$ .*

- (1)  *$\{t_w e_{\chi_\alpha} \mid w \in W_\alpha\}$  is an  $A$ -basis of  $\mathcal{H}_\alpha$  and so  $\mathcal{H}_\alpha$  is a free  $A$ -module with rank  $|W_\alpha|$ .*
- (2)  *$\{t_w e_{\chi_\alpha} \mid w \in W^\alpha\}$  is an  $\mathcal{H}_\alpha$ -basis of  $\mathcal{H}e_{\chi_\alpha}$  and so  $\mathcal{H}e_{\chi_\alpha}$  is a free  $\mathcal{H}_\alpha$ -module with rank  $|W^\alpha|$ .*

*Proof.* Because  $\{t_w e_\chi \mid w \in W, \chi \in X(D)\}$  is an  $A$ -basis of  $\mathcal{H}$ , the set  $\{e_{\chi_\alpha} t_w e_{\chi_\alpha} \mid w \in W, \chi \in X(D)\}$  spans  $\mathcal{H}_\alpha$ . If  $\chi \neq \chi_\alpha$ , then  $e_{\chi_\alpha} e_\chi = 0$  and so the set  $\{e_{\chi_\alpha} t_w e_{\chi_\alpha} \mid w \in W\}$  spans  $\mathcal{H}_\alpha$ . Since  $e_{\chi_\alpha} t_w e_{\chi_\alpha} = e_{\chi_\alpha} e_{w \cdot \chi_\alpha} t_w$ , it follows that  $e_{\chi_\alpha} t_w e_{\chi_\alpha} = 0$  if  $w \cdot \chi_\alpha \neq \chi_\alpha$ . If  $w \cdot \chi_\alpha = \chi_\alpha$ , then  $e_{\chi_\alpha} t_w e_{\chi_\alpha} = t_w e_{\chi_\alpha}$ . Thus, the set  $\{t_w e_{\chi_\alpha} \mid w \in W_\alpha\}$  spans  $\mathcal{H}_\alpha$ . Since this set is a subset of a basis of  $\mathcal{H}$ , it is linearly independent and so it is a basis of  $\mathcal{H}_\alpha$ .

Clearly  $\mathcal{H}e_{\chi_\alpha}$  is a free  $A$ -module with basis

$$\{t_w e_{\chi_\alpha} \mid w \in W\} = \{t_{w_1} e_{\chi_\alpha} (t_{w_2} e_{\chi_\alpha}) \mid w_1 \in W^\alpha, w_2 \in W_\alpha\}.$$

It follows from this observation and what has already been proved that the set  $\{t_w e_{\chi_\alpha} \mid w \in W^\alpha\}$  is an  $\mathcal{H}_\alpha$ -basis of  $\mathcal{H}e_{\chi_\alpha}$ .  $\square$

**3.13.** To continue, we need to make the isomorphism  $W_\alpha \cong \Sigma_{m_1} \times \dots \times \Sigma_{m_b}$  precise. Recall the integers  $\overline{m}_j$  and the sets  $M_j$  defined in terms of  $\alpha$  in 3.8. Suppose  $j \in [b]$  and  $\sigma \in \Sigma_{m_j}$ . Define a permutation  $w_\sigma^j \in W$  by

$$w_\sigma^j(k) = \begin{cases} k & \text{if } k \notin M_j \text{ and} \\ \overline{m}_{j-1} + \sigma(l) & \text{if } k = \overline{m}_{j-1} + l \in M_j. \end{cases}$$

Then the rule  $\sigma \mapsto w_\sigma^j$  defines an injective group homomorphism from  $\Sigma_{m_j}$  to  $W_\alpha$ . Let  $W_{\alpha,j}$  denote the image of this group homomorphism, so  $W_{\alpha,j} \cong \Sigma_{m_j}$ . Then  $W_\alpha = W_{\alpha,1} \cdots W_{\alpha,b}$  is the internal direct product of the subgroups  $W_{\alpha,j}$ .

In the following, we use two more notational conventions. First, because of the factor  $T_i^{(b^2-b)/2}$  in relation (r<sub>6</sub>), the parity of  $b$  will play a role in the rest of the proof of the paper. To simplify the notation, define  $\epsilon = \zeta^{(b^2-b)/2}$ . Then

$$\epsilon = 1 \text{ if } b \text{ is odd} \quad \text{and} \quad \epsilon = -1 \text{ if } b \text{ is even.}$$

Second, because  $\{M_1, \dots, M_b\}$  is a partition of the set  $[n]$ , for  $i \in [n]$ , there is a unique  $j \in [b]$  such that  $i \in M_j$ . Define  $i' = j$ . In other words,  $(\overline{m}_{j-1} + l)' = j$  for  $j \in [b]$  and  $l \in [m_j]$ .

With these conventions, for  $s_i \in W_\alpha$  define

$$\tilde{S}_i^\alpha = \epsilon^{i'} t_{s_i} e_{\chi_\alpha} \in \mathcal{H}_\alpha.$$

**Lemma 3.14.** *The elements  $\tilde{S}_i^\alpha$  for  $s_i \in W_\alpha$  generate the  $A$ -algebra  $\mathcal{H}_\alpha$  and satisfy the relations*

$$\begin{aligned} (b_1) \quad & \tilde{S}_{i_1}^\alpha \tilde{S}_{i_2}^\alpha = \tilde{S}_{i_2}^\alpha \tilde{S}_{i_1}^\alpha & s_{i_1}, s_{i_2} \in W_\alpha, \quad |i_1 - i_2| > 1, \\ (b_2) \quad & \tilde{S}_i^\alpha \tilde{S}_{i+1}^\alpha \tilde{S}_i^\alpha = \tilde{S}_{i+1}^\alpha \tilde{S}_i^\alpha \tilde{S}_{i+1}^\alpha & s_i, s_{i+1} \in W_\alpha, \quad \text{and} \\ (q) \quad & (\tilde{S}_i^\alpha)^2 = \mathbf{q} e_{\chi_\alpha} + \mathbf{b} a \tilde{S}_i^\alpha & s_i \in W_\alpha. \end{aligned}$$

*Proof.* It follows from Theorem 2.5 and equation 2.9(a) that the subalgebra generated by  $\{\tilde{S}_i^\alpha \mid s_i \in W_\alpha\}$  contains the basis  $\{t_w e_{\chi_\alpha} \mid w \in W_\alpha\}$  of  $\mathcal{H}_\alpha$  and hence is equal  $\mathcal{H}_\alpha$ .

Relations (b<sub>1</sub>) and (b<sub>2</sub>) follow immediately from relations (r<sub>4</sub>) and (r<sub>5</sub>) because  $t_{s_l}$  and  $e_{\chi_\alpha}$  commute when  $s_l \in W_\alpha$ .

To prove relation (q), using relation (r<sub>6</sub>) and that  $s_i \in W_{\chi_\alpha}$  we have

$$\begin{aligned} (\tilde{S}_i^\alpha)^2 &= (\epsilon^{i'} t_{s_i} e_{\chi_\alpha})^2 \\ &= t_{s_i}^2 e_{\chi_\alpha} \\ &= \left( \mathbf{q} + \mathbf{a} T_i^{(b^2-b)/2} \left( \sum_{k=0}^{b-1} T_i^k T_{i+1}^{-k} \right) R_i \right) e_{\chi_\alpha} \\ &= \mathbf{q} e_{\chi_\alpha} + \mathbf{a} T_i^{(b^2-b)/2} \left( \sum_{k=0}^{b-1} T_i^k T_{i+1}^{-k} \right) e_{\chi_\alpha} t_{s_i}. \end{aligned}$$

Notice that  $s_i \in W_\alpha$  implies that  $(i+1)' = i'$  and so by 3.1(a) we have

$$T_i^k T_{i+1}^{-k} e_{\chi_\alpha} = \zeta^{i'k+i'(-k)} e_{\chi_\alpha} = e_{\chi_\alpha} \quad \text{for } 0 \leq k \leq b-1$$

and

$$T_i^{(b^2-b)/2} e_{\chi_\alpha} = \zeta^{i'(b^2-b)/2} e_{\chi_\alpha} = \epsilon^{i'} e_{\chi_\alpha}.$$

Therefore,

$$(\tilde{S}_i^\alpha)^2 = \mathbf{q} e_{\chi_\alpha} + \epsilon^{i'} \mathbf{b} a e_{\chi_\alpha} t_{s_i} = \mathbf{q} e_{\chi_\alpha} + \mathbf{b} a \tilde{S}_i^\alpha.$$

□

The generators and relations in the lemma are in fact a presentation of  $\mathcal{H}_\alpha$ .

**Lemma 3.15.** *There is an  $A$ -algebra isomorphism*

$$\eta: \mathcal{I}^{m_1} \otimes_A \cdots \otimes_A \mathcal{I}^{m_b} \xrightarrow{\cong} \mathcal{H}_\alpha$$

with the property that if  $\rho_j \in \Sigma_{m_j}$  for  $j \in [b]$ , then  $\eta(S_{\rho_1} \otimes \cdots \otimes S_{\rho_b}) = \pm t_w e_{\chi_\alpha}$ , where  $w = w_{\rho_1}^1 \cdots w_{\rho_b}^b$ .



*Proof.* For  $j \in [b]$  let  $\mathcal{H}_{\alpha,j}$  be the subalgebra of  $\mathcal{H}_\alpha$  generated by the set  $\{\tilde{S}_i^\alpha \mid s_i \in W_{\alpha,j}\}$  and define

$$\eta_j: \mathcal{I}^{m_j} \rightarrow \mathcal{H}_\alpha \quad \text{by} \quad \eta_j(S_l) = \tilde{S}_{m_{j-1}+l}^\alpha = \epsilon^j t_{w_{\sigma_l}^j} e_{\chi_\alpha} \quad \text{for } l \in [m_j].$$

Then it follows from Lemma 3.12(1) and Lemma 3.14 that  $\eta_j$  induces an  $A$ -algebra isomorphism  $\mathcal{I}^{m_j} \cong \mathcal{H}_{\alpha,j}$ . If  $\rho \in \Sigma_{m_j}$  and  $\tau_{l_1} \cdots \tau_{l_r}$  is a reduced expression for  $\rho$ , then a straightforward computation shows that

$$(a) \quad \eta_j(S_\rho) = (\epsilon^j)^r t_{w_\rho^j} e_{\chi_\alpha}.$$

The homomorphisms  $\eta_j$  for  $j \in [b]$  determine an  $A$ -linear map

$$\eta: \mathcal{I}^{m_1} \otimes_A \cdots \otimes_A \mathcal{I}^{m_b} \rightarrow \mathcal{H}_\alpha \quad \text{with} \quad \eta(h_1 \otimes \cdots \otimes h_b) = \eta_1(h_1) \cdots \eta_b(h_b).$$

The map  $\eta$  is an  $A$ -algebra homomorphism because each  $\eta_j$  is an  $A$ -algebra homomorphism and because  $\mathcal{H}_{\alpha,j}$  and  $\mathcal{H}_{\alpha,j'}$  commute elementwise for  $j, j' \in [b]$  with  $j \neq j'$ . Because  $\eta$  is an  $A$ -algebra homomorphism, if  $\rho_j \in \Sigma_{m_j}$  for  $j \in [b]$  and  $w = w_{\rho_1}^1 \cdots w_{\rho_b}^b$ , then by (a)  $\eta(S_{\rho_1} \otimes \cdots \otimes S_{\rho_b}) = \pm t_w e_{\chi_\alpha}$ . Therefore,  $\eta$  maps the basis  $\{S_{\rho_1} \otimes \cdots \otimes S_{\rho_b} \mid \forall j \in [b], \rho_j \in \Sigma_{m_j}\}$  of  $\mathcal{I}^{m_1} \otimes_A \cdots \otimes_A \mathcal{I}^{m_b}$  bijectively onto a basis of  $\mathcal{H}_\alpha$ , and hence is an  $A$ -algebra isomorphism.  $\square$

**3.16. The  $\mathcal{H}$ -module  $V^\lambda$ .** Suppose  $\lambda = (\lambda^1, \dots, \lambda^b) \in b\mathcal{P}(n)$  and  $|\lambda| = \alpha$ . Recall from 3.10 that for  $j \in [b]$ ,  $P^{\lambda^j}$  is an  $\mathcal{I}^{m_j}$ -module with  $A$ -basis  $\{p_{\hat{\tau}} \mid \hat{\tau} \in \text{SYT}^{\lambda^j}\}$  and the action of the generator  $S_l \in \mathcal{I}^{m_j}$  on the basis element  $p_{\hat{\tau}}$  given by 3.10(a).

Define

$$P_0^\lambda = P^{\lambda^1} \otimes_A \cdots \otimes_A P^{\lambda^b}.$$

Then  $P_0^\lambda$  is an  $\mathcal{I}^{m_1} \otimes_A \cdots \otimes_A \mathcal{I}^{m_b}$ -module. We consider  $P_0^\lambda$  as an  $\mathcal{H}_\alpha$ -module via transport of structure across the isomorphism  $\eta: \mathcal{I}^{m_1} \otimes_A \cdots \otimes_A \mathcal{I}^{m_b} \xrightarrow{\cong} \mathcal{H}_\alpha$  in Lemma 3.15.

Finally, define

$$V^\lambda = F_\alpha(P_0^\lambda) = \mathcal{H} \otimes_{\mathcal{H}_\alpha} P_0^\lambda = \mathcal{H} e_{\chi_\alpha} \otimes_{\mathcal{H}_\alpha} P_0^\lambda.$$

We can now state the main theorem in this section.

**Theorem 3.17.** *Suppose  $\lambda \in b\mathcal{P}(n)$  and set  $\alpha = |\lambda| \in \mathcal{C}(n, b)$ . The  $\mathcal{H}$ -module  $V^\lambda$  has the following properties.*

- (1)  $V^\lambda$  is a free  $A$ -module with a basis  $\{v_\tau \mid \tau \in \text{SYT}^\lambda\}$  indexed by  $\text{SYT}^\lambda$ .
- (2) Suppose  $\tau \in \text{SYT}^\lambda$  and that  $\tau = w \cdot \tau_0$  with  $w \in W^\alpha$  and  $\tau_0 \in \text{SYT}_0^\lambda$ . Then  $v_\tau = t_w v_{\tau_0}$ .
- (3) For  $w \in W^\alpha$ , the weight space  $V_{w \cdot \chi_\alpha}^\lambda$  is the  $A$ -span of  $\{t_w v_{\tau_0} \mid \tau_0 \in \text{SYT}_0^\lambda\}$ . In particular,  $V_{\chi_\alpha}^\lambda$  is the  $A$ -span of  $\{v_{\tau_0} \mid \tau_0 \in \text{SYT}_0^\lambda\}$  and  $V_{\chi_\alpha}^\lambda \cong P_0^\lambda$ .
- (4)  $V^\lambda$  is in  $\mathcal{O}_\alpha$ .

(5) Suppose  $\tau \in SYT^\lambda$ . Then for  $d \in D$

$$t_d \cdot v_\tau = \chi_\tau(d) v_\tau,$$

and for  $i \in [n-1]$

$$t_{s_i} v_\tau = \begin{cases} v_{s_i \cdot \tau} & \text{if } j < j' \\ \epsilon^j \left( \frac{1-x}{y(1-x^k)} \right) v_\tau + \epsilon^j \left( \frac{x-x^k}{y(1-x^k)} \right) v_{s_i \cdot \tau} & \text{if } j = j' \\ \mathbf{q} v_{s_i \cdot \tau} & \text{if } j > j', \end{cases}$$

where  $j = \tau_i$ ,  $j' = \tau_{i+1}$ ,  $k = \rho_{\tau j}(i+1, i)$ , and  $v_{s_i \cdot \tau} = 0$  if  $s_i \cdot \tau$  is not standard.

(6) Let  $f: A \rightarrow k$  be a specialization. Then  ${}_f V^\lambda$  is an irreducible  ${}_f \mathcal{H}$ -module if and only if  ${}_f P_0^\lambda$  is an irreducible  ${}_f \mathcal{H}_\alpha$ -module.

*Proof.* By construction  $\{p_{\hat{\tau}^1} \otimes \cdots \otimes p_{\hat{\tau}^b} \mid \forall j \in [b], \hat{\tau}^j \in SYT^{\lambda^j}\}$  is an  $A$ -basis of  $P_0^\lambda$ . Given a tuple  $(\hat{\tau}^1, \dots, \hat{\tau}^b)$  with  $\hat{\tau}^j \in SYT^{\lambda^j}$  for  $j \in [b]$ , define  $\tau_0^j$  to be the tableau obtained from  $\hat{\tau}^j$  by adding  $\overline{m}_{j-1}$  to each entry and define  $\tau_0 = (\tau_0^1, \dots, \tau_0^b)$ . Then  $\tau_0 \in SYT_0^\lambda$ . Define

$$\tilde{v}_{\tau_0} = p_{\hat{\tau}^1} \otimes \cdots \otimes p_{\hat{\tau}^b}.$$

Then  $\{\tilde{v}_{\tau_0} \mid \tau_0 \in SYT_0^\lambda\}$  is an  $A$ -basis of  $P_0^\lambda$  and it follows from Lemma 3.12 (2) that  $\{t_w \otimes \tilde{v}_{\tau_0} \mid w \in W^\alpha, \tau_0 \in SYT_0^\lambda\}$  is an  $A$ -basis of  $V^\lambda$ . For  $w \in W^\alpha$  and  $\tau_0 \in SYT_0^\lambda$  define

$$v_\tau = t_w \otimes \tilde{v}_{\tau_0}.$$

Then  $\{v_\tau \mid \tau \in SYT^\lambda\}$  is an  $A$ -basis of  $V^\lambda$ . This proves (1). By definition,  $v_\tau = t_w \otimes \tilde{v}_{\tau_0} = t_w(1 \otimes \tilde{v}_{\tau_0}) = t_w v_{\tau_0}$  and so (2) holds as well.

Next, for  $\tau_0 \in SYT_0^\lambda$ ,  $e_{\chi_\alpha} v_{\tau_0} = e_{\chi_\alpha}(1 \otimes \tilde{v}_{\tau_0}) = 1 \otimes e_{\chi_\alpha} \tilde{v}_{\tau_0} = 1 \otimes \tilde{v}_{\tau_0} = v_{\tau_0}$ . This implies that  $v_{\tau_0} \in V_{\chi_\alpha}^\lambda$ . Therefore, it follows from 2.9(b) that  $t_w v_{\tau_0} \subseteq V_{w \cdot \chi_\alpha}^\lambda$  for every  $w \in W$ . In particular, if  $\tau_0 \in SYT_0^\lambda$  and  $w \in W^\alpha$ , then  $v_\tau = t_w v_{\tau_0} \in V_{w \cdot \chi_\alpha}^\lambda$ . Because  $\{v_\tau \mid \tau \in SYT^\lambda\}$  is a basis of  $V^\lambda$  this shows that for  $w \in W^\alpha$ ,  $V_{w \cdot \chi_\alpha}^\lambda$  is the  $A$ -span of  $\{t_w v_{\tau_0} \mid \tau_0 \in SYT_0^\lambda\}$ . In particular,  $V_{\chi_\alpha}^\lambda$  is the  $A$ -span of  $\{v_{\tau_0} \mid \tau_0 \in SYT_0^\lambda\}$  and the rule  $\tilde{v}_{\tau_0} \mapsto v_{\tau_0}$  defines an  $A$ -linear isomorphism between  $P_0^\lambda$  and  $V_{\chi_\alpha}^\lambda$ . This proves (3). In addition,  $V_\chi^\lambda = 0$  if  $\chi$  is not in the  $W$ -orbit of  $\chi_\alpha$ , and so  $V^\lambda$  is in  $\mathcal{O}_\alpha$  as asserted in (4).

To prove (5), fix  $\tau = w \cdot \tau_0$  in  $SYT^\lambda$  with  $w \in W^\alpha$  and  $\tau_0 \in SYT_0^\lambda$ . Then  $v_\tau \in V_{w \cdot \chi_\alpha}^\lambda$ , and so by 2.9(b), 3.7(a), and Lemma 3.6 we have

$$t_d \cdot v_\tau = (w \cdot \chi_\alpha)(d) \cdot v_\tau = (w \cdot \chi_{\tau_0})(d) \cdot v_\tau = \chi_{w \cdot \tau_0}(d) v_\tau = \chi_\tau(d) v_\tau.$$

Using the definitions and 3.10(a), a straightforward computation shows that for  $j \in [b]$  and  $s_l \in W_{\alpha, j}$  we have

$$(a) \quad t_{s_l} e_{\chi_\alpha} \cdot \tilde{v}_{\tau_0} = \eta^{-1}(t_{s_l} e_{\chi_\alpha}) \cdot \tilde{v}_{\tau_0} = \epsilon^j \left( \frac{x-1}{y(1-x^k)} \right) \tilde{v}_{\tau_0} + \epsilon^j \left( \frac{x-x^k}{y(1-x^k)} \right) \tilde{v}_{s_l \cdot \tau_0},$$

where  $k = \delta_{\tau_0^j}(l+1, )$  and  $\tilde{v}_{s_l \cdot \tau_0} = 0$  if  $s_l \cdot \tau_0$  is not standard.

Now suppose  $i \in [n-1]$  and consider  $t_{s_i} \cdot v_\tau = t_{s_i} t_w \cdot v_{\tau_0}$ . Suppose  $\tau_i = j$  and  $\tau_{i+1} = j'$ . Then  $i \in w(M_j)$  and  $i+1 \in w(M_{j'})$ . There are three cases. First, if  $j < j'$ , then  $s_i w > w$  and  $s_i w \in W^\alpha$ , so

$$t_{s_i} \cdot v_\tau = t_{s_i w} \cdot v_{\tau_0} = v_{s_i w \cdot \tau_0} = v_{s_i \cdot \tau}.$$

Second, if  $j = j'$ , then because  $w \in W^\alpha$  and  $i, i+1 \in w(M_j)$ , there is an  $l$  in  $M_j$  such that  $l < m_j$ ,  $w(l) = i$ , and  $w(l+1) = i+1$ . Then  $s_l \in W_{\alpha, j}$  and  $s_i w = w s_l$ . Therefore, using the definitions and (a) we have

$$\begin{aligned} t_{s_i} \cdot v_\tau &= t_w t_{s_l} \otimes \tilde{v}_{\tau_0} \\ &= t_w \otimes t_{s_l} e_{\chi_\alpha} \cdot \tilde{v}_{\tau_0} \\ &= t_w \otimes \epsilon^j \left( \frac{x-1}{y(1-x^k)} \tilde{v}_{\tau_0} + \frac{x-x^k}{y(1-x^k)} \tilde{v}_{s_l \cdot \tau_0} \right) \\ &= \epsilon^j \left( \frac{x-1}{y(1-x^k)} \right) t_w \otimes \tilde{v}_{\tau_0} + \epsilon^j \left( \frac{x-x^k}{y(1-x^k)} \right) t_w \otimes \tilde{v}_{s_l \cdot \tau_0} \\ &= \epsilon^j \left( \frac{x-1}{y(1-x^k)} \right) t_w \cdot v_{\tau_0} + \epsilon^j \left( \frac{x-x^k}{y(1-x^k)} \right) t_w v_{s_l \cdot \tau_0} \\ &= \epsilon^j \left( \frac{x-1}{y(1-x^k)} \right) v_\tau + \epsilon^j \left( \frac{x-x^k}{y(1-x^k)} \right) v_{s_i \cdot \tau}, \end{aligned}$$

where  $k = \delta_{\tau j}(i+1, i)$  and  $\tilde{v}_{s_l \cdot \tau_0} = 0 = v_{s_i \cdot \tau}$  if  $s_l \cdot \tau_0$ , or equivalently if  $s_i \cdot \tau$ , is not standard. Finally, if  $j > j'$ , then  $s_i w < w$  and  $s_i w \in W^\alpha$ . Therefore, by Theorem 2.5

$$\begin{aligned} t_{s_i} \cdot v_\tau &= t_{s_i} t_w \cdot v_{\tau_0} \\ &= (\mathbf{q} t_{s_i w} + \mathbf{a} T_i^{(b^2-b)/2} E_i t_w) \cdot v_{\tau_0} \\ &= \mathbf{q} t_{s_i w} \cdot v_{\tau_0} + \mathbf{a} T_i^{(b^2-b)/2} E_i t_w \cdot v_{\tau_0} \\ &= \mathbf{q} v_{s_i \cdot \tau} + \mathbf{a} T_i^{(b^2-b)/2} E_i t_w e_{\chi_\alpha} \cdot v_{\tau_0}. \end{aligned}$$

By Theorem 2.5 and 3.1(a) we have

$$\begin{aligned} E_i t_w e_{\chi_\alpha} &= \left( \sum_{r=0}^{b-1} T_i^r T_{i+1}^{-r} \right) t_w e_{\chi_\alpha} \\ &= t_w \left( \sum_{r=0}^{b-1} T_{w^{-1}(i)}^r T_{w^{-1}(i+1)}^{-r} \right) e_{\chi_\alpha} \\ &= \left( \sum_{r=0}^{b-1} (\zeta^{j-j'})^r \right) t_w e_{\chi_\alpha} = 0 \end{aligned}$$

because  $j \neq j'$  and so  $\zeta^{j-j'}$  is a  $b^{\text{th}}$  root of unity not equal to 1. Therefore,

$$t_{s_i} \cdot v_\tau = \mathbf{q} v_{s_i \cdot \tau}.$$

The last statement in the theorem follows from Theorem 2.11 and the isomorphism  ${}_f P_0^\lambda \cong {}_f V_{\chi_\alpha}^\lambda$  that follows from the third statement of the theorem.  $\square$

### 3.18. Irreducible modules and semisimple specializations.

**Corollary 3.19.** *Let  $f: A \rightarrow k$  be a specialization such that  $k$  is a field and consider the specialized algebra  ${}_f\mathcal{H}$ .*

- (1) *Suppose that  $\lambda = (\lambda^1, \dots, \lambda^b) \in b\mathcal{P}(n)$  and that for all  $j \in [b]$  the  ${}_f\mathcal{I}^{|\lambda^j|}$ -module  ${}_fP^{\lambda^j}$  is irreducible. Then the  ${}_f\mathcal{H}$ -module  ${}_fV^\lambda$  is absolutely irreducible.*
- (2) *Suppose that  $\alpha = (m_1, \dots, m_b) \in \mathcal{C}(n, b)$  and that for all  $\lambda = (\lambda^1, \dots, \lambda^b) \in b\mathcal{P}(n)$  with  $|\lambda| = \alpha$  and all  $j \in [b]$ , the  ${}_f\mathcal{I}^{m_j}$ -module  ${}_fP^{\lambda^j}$  is irreducible. Then  ${}_f\mathcal{H}_\alpha$  is a split semisimple  $k$ -algebra and  $\{{}_fV^\lambda \mid \lambda \in b\mathcal{P}(n), |\lambda| = \alpha\}$  is a complete set of inequivalent, irreducible  ${}_f\mathcal{H}$ -modules in  ${}_f\mathcal{O}_\alpha$ .*
- (3) *Suppose that for all  $\lambda = (\lambda^1, \dots, \lambda^b) \in b\mathcal{P}(n)$  and all  $j \in [b]$ , the  ${}_f\mathcal{I}^{|\lambda^j|}$ -module  ${}_fP^{\lambda^j}$  is irreducible. Then  ${}_f\mathcal{H}$  is a split semisimple  $k$ -algebra and  $\{{}_fV^\lambda \mid \lambda \in b\mathcal{P}(n)\}$  is a complete set of inequivalent, irreducible  ${}_f\mathcal{H}$ -modules.*

*Proof.* The algebra  $\mathcal{I}^m$  is a cellular algebra in the sense of Graham and Lehrer and  $P^{\hat{\lambda}}$  is a cell module for every partition  $\hat{\lambda}$  of  $m$ . Thus, it follows from [6] that if  ${}_fP^{\hat{\lambda}}$  is irreducible, then it is absolutely irreducible. It then follows that if  ${}_fP^{\hat{\lambda}}$  is irreducible for all  $\hat{\lambda} \in \mathcal{P}(m)$ , then  ${}_f\mathcal{I}^m$  is a split semisimple  $k$ -algebra.

To prove the first statement, set  $\alpha = |\lambda|$ . By assumption,  ${}_fP^{\lambda^j}$  is irreducible for  $j \in [b]$ , so  ${}_fP_0^\lambda \cong {}_fP^{\lambda^1} \otimes_k \dots \otimes_k {}_fP^{\lambda^b}$  is a product of absolutely irreducible modules and hence is absolutely irreducible. It follows that  ${}_fV^\lambda$  is absolutely irreducible because induction from  ${}_f\mathcal{H}_\alpha$ -mod to  ${}_f\mathcal{O}_\alpha$  is an equivalence of categories by Theorem 2.11.

Now suppose  $\alpha = (m_1, \dots, m_b) \in \mathcal{C}(n, b)$  and that for all  $\lambda = (\lambda^1, \dots, \lambda^b) \in b\mathcal{P}(n)$  with  $|\lambda| = \alpha$  and all  $j \in [b]$ , the  ${}_f\mathcal{I}^{m_j}$ -module  ${}_fP^{\lambda^j}$  is absolutely irreducible. Then for all  $j \in [b]$ ,  ${}_f\mathcal{I}^{m_j}$  is a split semisimple  $k$ -algebra with simple modules  $\{{}_fP^{\hat{\lambda}} \mid \hat{\lambda} \in \mathcal{P}(m_j)\}$ . Thus, it follows from Lemma 3.15 that  ${}_f\mathcal{H}_\alpha$  is split semisimple and that the set  $\{{}_fP_0^\lambda \mid \lambda \in b\mathcal{P}(n), |\lambda| = \alpha\}$  is a complete set of inequivalent, irreducible  ${}_f\mathcal{H}_\alpha$ -modules. Therefore, by Theorem 2.11, the set  $\{{}_fV^\lambda \mid \lambda \in b\mathcal{P}(n), |\lambda| = \alpha\}$  is a complete set of inequivalent, irreducible  ${}_f\mathcal{H}$ -modules in  ${}_f\mathcal{O}_\alpha$ . This proves the second statement.

Finally, suppose that for all  $\lambda = (\lambda^1, \dots, \lambda^b) \in b\mathcal{P}(n)$  and all  $j \in [b]$ , the  ${}_f\mathcal{I}^{m_j}$ -module  ${}_fP^{\lambda^j}$  is absolutely irreducible. Then it follows from the block decomposition 2.10(a) and what has already been proved that the set  $\{{}_fV^\lambda \mid \lambda \in b\mathcal{P}(n)\}$  is a complete set of inequivalent, irreducible  ${}_f\mathcal{H}$ -modules, each of which is absolutely irreducible. Therefore

$$\dim({}_f\mathcal{H}/\text{rad } {}_f\mathcal{H}) = \sum_{\lambda \in b\mathcal{P}(n)} (\dim {}_fV^\lambda)^2 = \sum_{\lambda \in b\mathcal{P}(n)} |\text{SYT}^\lambda|^2.$$

But  $\sum_{\lambda \in b\mathcal{P}(n)} |\text{SYT}^\lambda|^2 = b^n n! = \dim {}_f\mathcal{H}$ , and so  $\text{rad } {}_f\mathcal{H} = 0$ , which implies that  ${}_f\mathcal{H}$  is a split semisimple  $k$ -algebra.  $\square$

Recall that  $K$  is the quotient field of  $A$ . Hoefsmit [7] has shown that for  $m > 1$  and  $\hat{\lambda} \in \mathcal{P}(m)$ , the  ${}_K\mathcal{I}^m$ -module  ${}_KP^{\hat{\lambda}}$  is irreducible, and so the next corollary follows

from Corollary 3.19(3).

**Corollary 3.20.** *The  $K$ -algebra  ${}_K\mathcal{H}$  is split semisimple and  $\{ {}_KV^\lambda \mid \lambda \in b\mathcal{P}(n) \}$  is a complete set of inequivalent, irreducible  ${}_K\mathcal{H}$ -modules.*

#### 4. CONNECTIONS WITH YOKONUMA-HECKE ALGEBRAS

In this section we explain the connections between the constructions and results in this paper and the related constructions and results of Thiem [12], Juyumaya [9] [10], Chlouveraki and Poulain d’Andecy [3], and Jacon and Poulain d’Andecy [8].

Suppose  $q$  is a prime power and  $f: \mathbb{Z}[\mathbf{q}, \mathbf{a}] \rightarrow \mathbb{C}$  is the specialization with  $f(\mathbf{q}) = q$  and  $f(\mathbf{a}) = 1$ . Then as recalled in 2.6, the specialized algebra  ${}_f\mathcal{H}_{q-1,n}$  is isomorphic to the Hecke algebra  $\mathcal{H}_{\mathbb{C}}(G, U)$ .

**4.1. Thiem’s thesis.** This paper is an extension of Thiem’s study of  $\mathcal{H}_{\mathbb{C}}(G, U)$  in [12, Chapter 6] to the more general case of Yokonuma-type Hecke algebras. The main differences are the following.

- (1) This paper deals with a larger class of algebras that are “generic” algebras with minimal assumptions on the underlying ring of scalars. The results in [12, Chapter 6] are obtained from the results in this paper by taking  $b = q - 1$  and setting  $\mathbf{q} = q$  and  $\mathbf{a} = 1$ .
- (2) The formulation of the block/weight space decomposition of  $\mathcal{H}$  is formalized and placed in a categorical framework.
- (3) More details of the structure of the representations  $V^\lambda$  in §3 are worked out.

**4.2. Juyumaya’s presentation.** Juyumaya [10] found an intriguing presentation of the Hecke algebra  $\mathcal{H}_{\mathbb{C}}(G, U)$  that leads to a simpler presentation for one-parameter Yokonuma-Hecke algebras than the presentation arising from 2.1 when  $b = q - 1$ . Using the notation above, Juyumaya’s presentation is described as follows.

Suppose for the moment that  $q$  is a power of an odd prime  $p$  and  $\xi$  is a primitive  $p^{\text{th}}$  root of unity in  $\mathbb{C}$ . Let  $h: \mathbb{Z}[\mathbf{q}, \mathbf{a}] \rightarrow \mathbb{Z}[\xi][\mathbf{q}, \mathbf{q}^{-1}]$  be the specialization with  $h(\mathbf{q}) = \mathbf{q}$  and  $h(\mathbf{a}) = 1$  and consider the specialized algebra  ${}_h\mathcal{H}_{q-1,n}$ . To streamline the notation, set  $\tilde{A} = \mathbb{Z}[\xi][\mathbf{q}, \mathbf{q}^{-1}]$ ,  $\tilde{\mathcal{H}} = {}_h\mathcal{H}_{q-1,n}$ , and denote the generators  $1 \otimes T_j$  and  $1 \otimes R_i$  of  $\tilde{\mathcal{H}}$  simply by  $T_j$  and  $R_i$ , respectively. With these conventions, the quadratic relation (r6) becomes

$$R_i^2 = \mathbf{q} + T_i^{(q-1)/2} E_i R_i \quad \text{for } i \in [n-1].$$

Fix a generator  $\omega$  of  $\mathbb{F}_q^\times$  and a non-trivial character  $\psi: \mathbb{F}_q \rightarrow \tilde{A}^\times$  of the additive group of  $\mathbb{F}_q$ . For  $i \in [n-1]$  define

$$F_i = \sum_{j=1}^{q-1} \psi(\omega^j) T_i^j, \quad \text{and} \quad R'_i = T_i^{(q-1)/2} R_i \quad \text{in } \tilde{\mathcal{H}}.$$

Juyumaya’s presentation is in terms of the elements

$$J_i = q^{-1}(E_i + R'_i F_i) \quad \text{and} \quad P_i = (q^2 - 1)^{-1}(E_i + R'_i E_i).$$

The arguments in [10, §2] show that  $\tilde{\mathcal{H}}$  has generators

$$T_1, \dots, T_n, J_1, \dots, J_{n-1}$$

and relations  $(\mathbf{r}_1)$ ,  $(\mathbf{r}_2)$ , and

$$\begin{aligned} (\text{jr}_3) \quad & T_j J_i = J_i T_{s_i(j)} && \text{for } j \in [n] \text{ and } i \in [n-1], \\ (\text{jr}_4) \quad & J_i J_{i'} = J_{i'} J_i && \text{for } i, i' \in [n-1] \text{ with } |i - i'| > 1, \\ (\text{jr}_5) \quad & J_i J_{i+1} J_i = J_{i+1} J_i J_{i+1} && \text{for } i \in [n-2], \text{ and} \\ (\text{jr}_6) \quad & J_i^2 = 1 + (\mathbf{q}^{-2} - 1)P_i && \text{for } i \in [n-1]. \end{aligned}$$

This is essentially the presentation in [9, Theorem 3] and [10, Theorem 2.18].

To get a presentation similar to that in 2.1, for  $i \in [n-1]$  define

$$G_i = -\mathbf{q}^{-1} R'_i (F_i - (\mathbf{v} + 1)^{-1} E_i),$$

where  $\mathbf{v}$  is a square root of  $\mathbf{q}$ . It can be shown that  $\tilde{\mathcal{H}}$  has a presentation with generators

$$T_1, \dots, T_n, G_1, \dots, G_{n-1}$$

and relations  $(\mathbf{r}_1)$ ,  $(\mathbf{r}_2)$ , and

$$\begin{aligned} (\text{gr}_3) \quad & T_j G_i = G_i T_{s_i(j)} && \text{for } j \in [n] \text{ and } i \in [n-1], \\ (\text{gr}_4) \quad & G_i G_{i'} = G_{i'} G_i && \text{for } i, i' \in [n-1] \text{ with } |i - i'| > 1, \\ (\text{gr}_5) \quad & G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1} && \text{for } i \in [n-2], \text{ and} \\ (\text{gr}_6) \quad & G_i^2 = \mathbf{q} + E_i G_i && \text{for } i \in [n-1]. \end{aligned}$$

Notice that if  $q$  is a power of two, then relation  $(\mathbf{r}_6)$  in  ${}_f\mathcal{H}_{q-1,n}$  is simply

$$R_i^2 = q + E_i R_i \quad \text{for } i \in [n-1],$$

and so the presentation just given for the one-parameter Yokonuma-Hecke algebra  $\tilde{\mathcal{H}} = {}_h\mathcal{H}_{q-1,n}$  when  $q$  is odd is valid for all prime powers.

Because  $G_i$  is defined in terms of  $F_i$  and  $F_i$  depends on the additive character  $\psi$ , and thus on the arithmetic of the field  $\mathbb{F}_q$ , it is not immediately clear how to modify Juyumaya's construction to obtain a presentation of the Yokonuma-type Hecke algebra  $\mathcal{H}_{b,n}$  when  $b \neq q-1$ .

**4.3. Work of Chlouveraki, Jacon, and Poulain d'Andecy.** Suppose as above that  $\mathbf{v}$  is a square root of  $\mathbf{q}$ . Jacon and Poulain d'Andecy [8] define two-parameter Yokonuma-Hecke algebras  $Y_{b,n}$  that are  $\mathbb{C}[\mathbf{v}, \mathbf{v}^{-1}, \mathbf{a}]$ -algebras with generators

$$T_1, \dots, T_n, G_1, \dots, G_{n-1}$$

and relations  $(\mathbf{r}_1)$ ,  $(\mathbf{r}_2)$ ,  $(\text{gr}_3)$ ,  $(\text{gr}_4)$ ,  $(\text{gr}_5)$ , and the quadratic relation

$$(\text{gr}'_6) \quad G_i^2 = \mathbf{v}^2 + \mathbf{a}(b^{-1} E_i) G_i \quad \text{for } i \in [n-1].$$

**Lemma 4.4.** *Consider the specialization  $g: \mathbb{Z}[\mathbf{q}, \mathbf{a}] \rightarrow \mathbb{C}[\mathbf{v}, \mathbf{v}^{-1}, \mathbf{a}]$  with  $g(\mathbf{q}) = \mathbf{v}^2$  and  $g(\mathbf{a}) = b^{-1}\mathbf{a}$ . Then  ${}_g\mathcal{H}_{b,n} = Y_{b,n}$ .*

*Proof.* The result is clear when  $b$  is odd.

Suppose now that  $b$  is even and for  $i \in [n-1]$  define

$$u_i = \sum_{\substack{\chi \in X(D) \\ s_i \cdot \chi = \chi}} T_i^{b/2} e_\chi + \sum_{\substack{\chi \in X(D) \\ s_i \cdot \chi \neq \chi}} e_\chi \quad \text{and} \quad \tilde{R}_i = u_i R_i.$$

Using the fact that

$$\sum_{j=0}^{b-1} T_i^j T_{i+1}^{-j} e_\chi = \begin{cases} b e_\chi & \text{if } s_i \cdot \chi = \chi \\ 0 & \text{if } s_i \cdot \chi \neq \chi \end{cases}$$

it is straightforward to check that  $u_i^2 = 1$ ,  $u_i R_i = R_i u_i$ , and  $T_i^{(b^2-b)/2} E_i u_i = E_i$ , and hence that  $\mathcal{H}_{b,n}$  has generators

$$T_1, \dots, T_n, \tilde{R}_1, \dots, \tilde{R}_{n-1}$$

and relations  $(\mathbf{r}_1)$ ,  $(\mathbf{r}_2)$ ,  $(\mathbf{gr}_3)$ ,  $(\mathbf{gr}_4)$ ,  $(\mathbf{gr}_5)$ , and the quadratic relation

$$(\mathbf{gr}_{6t}) \quad \tilde{R}_i^2 = \mathbf{q} + \mathbf{a} E_i \tilde{R}_i \quad \text{for } i \in [n-1].$$

It follows that  ${}_g\mathcal{H}_{b,n} = Y_{b,n}$  in this case as well.  $\square$

Chlouveraki and Poulain d'Andecy [3] define one-parameter Yokonuma-Hecke algebras that we denote by  $Y'_{b,n}$ . These are  $\mathbb{C}[\mathbf{v}, \mathbf{v}^{-1}]$ -algebras with generators

$$T_1, \dots, T_n, G_1, \dots, G_{n-1}$$

and relations  $(\mathbf{r}_1)$ ,  $(\mathbf{r}_2)$ ,  $(\mathbf{gr}_3)$ ,  $(\mathbf{gr}_4)$ ,  $(\mathbf{gr}_5)$ , and the quadratic relation

$$(\mathbf{gr}_6'') \quad G_i^2 = \mathbf{v}^2 + (\mathbf{v}^2 - 1)(b^{-1} E_i) G_i \quad \text{for } i \in [n-1].$$

Setting  $\tilde{G}_i = \mathbf{v}^{-1} G_i$  for  $i \in [n-1]$  gives the presentation in [3, 2.1] with quadratic relation

$$\tilde{G}_i^2 = 1 + (\mathbf{v} - \mathbf{v}^{-1})(b^{-1} E_i) \tilde{G}_i.$$

It follows from Lemma 4.4 that if  $g': \mathbb{Z}[\mathbf{q}, \mathbf{a}] \rightarrow \mathbb{C}[\mathbf{v}, \mathbf{v}^{-1}]$  is the specialization with  $g'(\mathbf{q}) = \mathbf{v}^2$  and  $g'(\mathbf{a}) = b^{-1}(\mathbf{v}^2 - 1)$ , then  $Y'_{b,n} = {}_{g'}\mathcal{H}_{b,n}$ . If  $q$  is a prime power and  $f': \mathbb{C}[\mathbf{v}, \mathbf{v}^{-1}] \rightarrow \mathbb{C}$  with  $f'(\mathbf{v}) = \sqrt{q}$ , then the quadratic relation  $(\mathbf{gr}_6'')$  simplifies to the relation  $(\mathbf{gr}_6)$  and so  ${}_{f'}Y'_{q-1,n} = {}_f\mathcal{H}_{q-1,n} \cong \mathcal{H}_{\mathbb{C}}(G, U)$ .

A family of  $Y'_{b,n}$ -modules indexed by  $b\mathcal{P}(n)$  is constructed in [3, §5.1] by defining an action of the generators  $T_j$  and  $G_i$  on basis vectors and checking that the defining relations of  $Y'_{b,n}$  hold. For  $\lambda \in b\mathcal{P}(n)$ , denote the module constructed in [3, §5.1] by  $\overline{V}^\lambda$ . With the notation in [3, §5.1],  $\{\mathbf{v}_\mathcal{T} \mid \mathcal{T} \in \text{SYT}^\lambda\}$  is a basis of  $\overline{V}^\lambda$ . It is straightforward to check that the obvious bijection between basis elements between  $\overline{V}^\lambda$  and  ${}_gV^\lambda$ , namely  $\mathbf{v}_\mathcal{T} \leftrightarrow 1 \otimes v_\mathcal{T}$ , defines an isomorphism of  $Y'_{b,n} = {}_{g'}\mathcal{H}_{b,n}$ -modules  $\overline{V}^\lambda \cong {}_gV^\lambda$ . Thus it follows from Theorem 3.17 that each  $\overline{V}^\lambda$  is an induced module.

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